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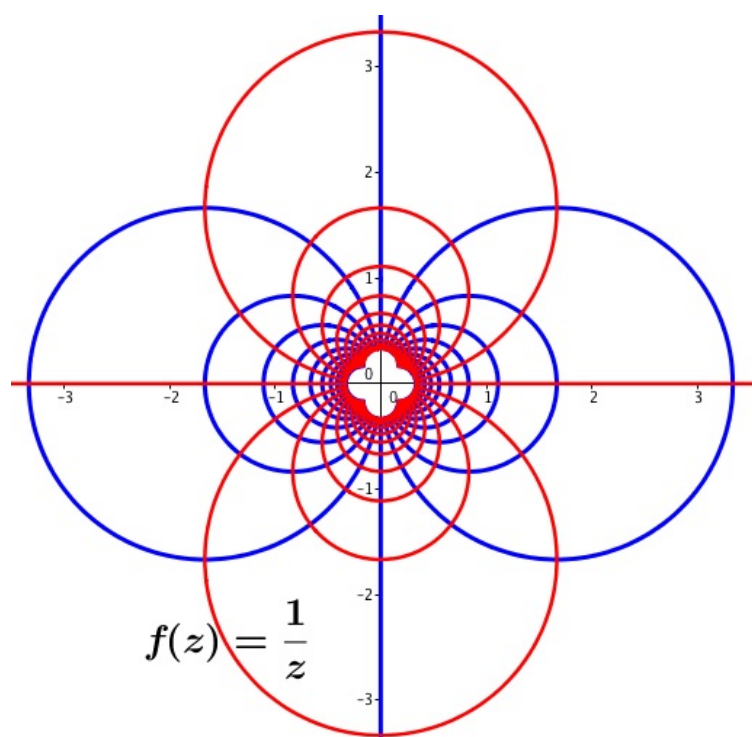
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# COMPLEX ANALYSIS

Problems with solutions



Juan Carlos Ponce Campuzano



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# Proem

This text constitutes a collection of problems for using as an additional learning resource for those who are taking an introductory course in complex analysis. The problems are numbered and allocated in four chapters corresponding to different subject areas: *Complex Numbers*, *Functions*, *Complex Integrals* and *Series*. The majority of problems are provided with answers, detailed procedures and hints (sometimes incomplete solutions).

Of course, no project such as this can be free from errors and incompleteness. I will be grateful to everyone who points out any typos, incorrect solutions, or sends any other suggestion for improving this manuscript.

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2015



# Chapter 1

## Complex Numbers

### 1.1 Basic algebraic properties

1. Verify that

$$(a) \left( \sqrt{2} - i \right) - i \left( 1 - \sqrt{2}i \right) = -2i$$

$$(b) (2 - 3i)(-2 + i) = -1 + 8i$$

*Solution.* We have

$$\left( \sqrt{2} - i \right) - i \left( 1 - \sqrt{2}i \right) = \sqrt{2} - i - i + \sqrt{2} = -2i,$$

and

$$(2 - 3i)(-2 + i) = -4 + 2i + 6i - 3i^2 = -4 + 3 + 8i = -1 + 8i.$$

■

2. Reduce the quantity

$$\frac{5i}{(1 - i)(2 - i)(3 - i)}$$

to a real number.

*Solution.* We have

$$\frac{5i}{(1 - i)(2 - i)(3 - i)} = \frac{5i}{(1 - i)(5 - 5i)} = \frac{i}{(1 - i)^2} = \frac{i}{-2i} = \frac{1}{2}$$

■



3. Show that

$$(a) \operatorname{Re}(iz) = -\operatorname{Im}(z);$$

$$(b) \operatorname{Im}(iz) = \operatorname{Re}(z).$$

*Proof.* Let  $z = x + yi$  with  $x = \operatorname{Re}(z)$  and  $y = \operatorname{Im}(z)$ . Then

$$\operatorname{Re}(iz) = \operatorname{Re}(-y + xi) = -y = -\operatorname{Im}(z)$$

and

$$\operatorname{Im}(iz) = \operatorname{Im}(-y + xi) = x = \operatorname{Re}(z).$$

■

4. Verify the associative law for multiplication of complex numbers. That is, show that

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

for all  $z_1, z_2, z_3 \in \mathbb{C}$ .

*Proof.* Let  $z_k = x_k + iy_k$  for  $k = 1, 2, 3$ . Then

$$\begin{aligned} (z_1 z_2) z_3 &= ((x_1 + y_1 i)(x_2 + y_2 i))(x_3 + y_3 i) \\ &= ((x_1 x_2 - y_1 y_2) + i(x_2 y_1 + x_1 y_2))(x_3 + y_3 i) \\ &= (x_1 x_2 x_3 - x_3 y_1 y_2 - x_2 y_1 y_3 - x_1 y_2 y_3) \\ &\quad + i(x_2 x_3 y_1 + x_1 x_3 y_2 + x_1 x_2 y_3 - y_1 y_2 y_3) \end{aligned}$$

and

$$\begin{aligned} z_1 (z_2 z_3) &= (x_1 + y_1 i)((x_2 + y_2 i)(x_3 + y_3 i)) \\ &= (x_1 + y_1 i)((x_2 x_3 - y_2 y_3) + i(x_2 y_3 + x_3 y_2)) \\ &= (x_1 x_2 x_3 - x_3 y_1 y_2 - x_2 y_1 y_3 - x_1 y_2 y_3) \\ &\quad + i(x_2 x_3 y_1 + x_1 x_3 y_2 + x_1 x_2 y_3 - y_1 y_2 y_3) \end{aligned}$$

Therefore,

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

■

5. Compute

- (a)  $\frac{2+i}{2-i}$ ;  
 (b)  $(1-2i)^4$ .

*Answer:* (a)  $(3+4i)/5$ , (b)  $-7+24i$ .

6. Let  $f$  be the map sending each complex number

$$z = x + yi \rightarrow \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

Show that  $f(z_1 z_2) = f(z_1) f(z_2)$  for all  $z_1, z_2 \in \mathbb{C}$ .

*Proof.* Let  $z_k = x_k + y_k i$  for  $k = 1, 2$ . Then

$$z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + i(x_2 y_1 + x_1 y_2)$$

and hence

$$f(z_1 z_2) = \begin{bmatrix} x_1 x_2 - y_1 y_2 & x_2 y_1 + x_1 y_2 \\ -x_2 y_1 - x_1 y_2 & x_1 x_2 - y_1 y_2 \end{bmatrix}.$$

On the other hand,

$$f(z_1) f(z_2) = \begin{bmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 x_2 - y_1 y_2 & x_2 y_1 + x_1 y_2 \\ -x_2 y_1 - x_1 y_2 & x_1 x_2 - y_1 y_2 \end{bmatrix}.$$

Therefore,  $f(z_1 z_2) = f(z_1) f(z_2)$ . ■

7. Use binomial theorem

$$\begin{aligned} (a+b)^n &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \dots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n \\ &= \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k \end{aligned}$$

to expand

- (a)  $(1 + \sqrt{3}i)^{2011}$ ;  
 (b)  $(1 + \sqrt{3}i)^{-2011}$ .

*Solution.* By binomial theorem,

$$(1 + \sqrt{3}i)^{2011} = \sum_{k=0}^{2011} \binom{2011}{k} (\sqrt{3}i)^k = \sum_{k=0}^{2011} \binom{2011}{k} 3^{k/2} i^k.$$

Since  $i^k = (-1)^m$  for  $k = 2m$  even and  $i^k = (-1)^m i$  for  $k = 2m + 1$  odd,

$$\begin{aligned} (1 + \sqrt{3}i)^{2011} &= \sum_{0 \leq 2m \leq 2011} \binom{2011}{2m} 3^m (-1)^m \\ &\quad + i \sum_{0 \leq 2m+1 \leq 2011} \binom{2011}{2m+1} 3^m \sqrt{3} (-1)^m \\ &= \sum_{m=0}^{1005} \binom{2011}{2m} (-3)^m + i \sum_{m=0}^{1005} \binom{2011}{2m+1} (-3)^m \sqrt{3}. \end{aligned}$$

Similarly,

$$\begin{aligned} (1 + \sqrt{3}i)^{-2011} &= \left( \frac{1}{1 + \sqrt{3}i} \right)^{2011} = \left( \frac{1 - \sqrt{3}i}{4} \right)^{2011} \\ &= \frac{1}{4^{2011}} \sum_{k=0}^{2011} \binom{2011}{k} (-\sqrt{3}i)^k \\ &= \frac{1}{4^{2011}} \sum_{m=0}^{1005} \binom{2011}{2m} (-3)^m \\ &\quad - \frac{i}{4^{2011}} \sum_{m=0}^{1005} \binom{2011}{2m+1} (-3)^m \sqrt{3}. \end{aligned}$$

■

## 1.2 Modulus

1. Show that

$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

for all  $z_1, z_2 \in \mathbb{C}$ .

*Proof.* We have

$$\begin{aligned}
 & |z_1 - z_2|^2 + |z_1 + z_2|^2 \\
 &= (z_1 - z_2)\overline{(z_1 - z_2)} + (z_1 + z_2)\overline{(z_1 + z_2)} \\
 &= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) + (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\
 &= ((z_1\bar{z}_1 + z_2\bar{z}_2) - (z_1\bar{z}_2 + z_2\bar{z}_1)) + ((z_1\bar{z}_1 + z_2\bar{z}_2) + (z_1\bar{z}_2 + z_2\bar{z}_1)) \\
 &= 2(z_1\bar{z}_1 + z_2\bar{z}_2) = 2(|z_1|^2 + |z_2|^2).
 \end{aligned}$$

■

2. Verify that  $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$ .

**Hint:** Reduce this inequality to  $(|x| - |y|)^2 \geq 0$ .

*Solution.* Note that

$$0 \leq (|\operatorname{Re} z| + |\operatorname{Im} z|)^2 = |\operatorname{Re} z|^2 + 2|\operatorname{Re} z||\operatorname{Im} z| + |\operatorname{Im} z|^2.$$

Thus

$$2|\operatorname{Re} z||\operatorname{Im} z| \leq |\operatorname{Re} z|^2 + |\operatorname{Im} z|^2,$$

and then

$$|\operatorname{Re} z|^2 + 2|\operatorname{Re} z||\operatorname{Im} z| + |\operatorname{Im} z|^2 \leq 2(|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2).$$

That is

$$(|\operatorname{Re} z| + |\operatorname{Im} z|)^2 \leq 2(|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2) = 2|z|^2,$$

and therefore,

$$|\operatorname{Re} z| + |\operatorname{Im} z| \leq \sqrt{2}|z|.$$

■

3. Sketch the curves in the complex plane given by

(a)  $\operatorname{Im}(z) = -1$ ;

(b)  $|z - 1| = |z + i|$ ;

(c)  $2|z| = |z - 2|$ .

*Solution.* Let  $z = x + yi$ .

(a)  $\{\operatorname{Im}(z) = -1\} = \{y = -1\}$  is the horizontal line passing through the point  $-i$ .

(b) Since

$$\begin{aligned} |z - 1| = |z + i| &\Leftrightarrow |(x - 1) + yi| = |x + (y + 1)i| \\ &\Leftrightarrow |(x - 1) + yi|^2 = |x + (y + 1)i|^2 \\ &\Leftrightarrow (x - 1)^2 + y^2 = x^2 + (y + 1)^2 \\ &\Leftrightarrow x + y = 0, \end{aligned}$$

the curve is the line  $x + y = 0$ .

(c) Since

$$\begin{aligned} 2|z| = |z - 2| &\Leftrightarrow 2|x + yi| = |(x - 2) + yi| \\ &\Leftrightarrow 4|x + yi|^2 = |(x - 2) + yi|^2 \\ &\Leftrightarrow 4(x^2 + y^2) = (x - 2)^2 + y^2 \\ &\Leftrightarrow 3x^2 + 4x + 3y^2 = 4 \\ &\Leftrightarrow \left(x + \frac{2}{3}\right)^2 + y^2 = \frac{16}{9} \\ &\Leftrightarrow \left|z + \frac{2}{3}\right| = \frac{4}{3} \end{aligned}$$

the curve is the circle with centre at  $-2/3$  and radius  $4/3$ .

■

4. Show that

$$\frac{R^4 - R}{R^2 + R + 1} \leq \left| \frac{z^4 + iz}{z^2 + z + 1} \right| \leq \frac{R^4 + R}{(R - 1)^2}$$

for all  $z$  satisfying  $|z| = R > 1$ .

*Proof.* When  $|z| = R > 1$ ,

$$|z^4 + iz| \geq |z^4| - |iz| = |z|^4 - |i||z| = R^4 - R$$

and

$$|z^2 + z + 1| \leq |z^2| + |z| + |1| = |z|^2 + |z| + 1 = R^2 + R + 1$$

by triangle inequality. Hence

$$\left| \frac{z^4 + iz}{z^2 + z + 1} \right| \geq \frac{R^4 - R}{R^2 + R + 1}.$$

On the other hand,

$$|z^4 + iz| \leq |z^4| + |iz| = |z|^4 + |i||z| = R^4 + R$$

and

$$\begin{aligned} |z^2 + z + 1| &= \left| \left( z - \frac{-1 + \sqrt{3}i}{2} \right) \left( z - \frac{-1 - \sqrt{3}i}{2} \right) \right| \\ &= \left| z - \frac{-1 + \sqrt{3}i}{2} \right| \left| z - \frac{-1 - \sqrt{3}i}{2} \right| \\ &\geq \left( |z| - \left| \frac{-1 + \sqrt{3}i}{2} \right| \right) \left( |z| - \left| \frac{-1 - \sqrt{3}i}{2} \right| \right) \\ &= (R - 1)(R - 1) = (R - 1)^2 \end{aligned}$$

Therefore,

$$\left| \frac{z^4 + iz}{z^2 + z + 1} \right| \leq \frac{R^4 + R}{(R - 1)^2}.$$

■

5. Use complex numbers to prove the **Law of Cosine**: Let  $\triangle ABC$  be a triangle with  $|BC| = a$ ,  $|CA| = b$ ,  $|AB| = c$  and  $\angle BCA = \theta$ . Then

$$a^2 + b^2 - 2ab \cos \theta = c^2.$$

**Hint:** Place  $C$  at the origin,  $B$  at  $z_1$  and  $A$  at  $z_2$ . Prove that

$$z_1 \bar{z}_2 + z_2 \bar{z}_1 = 2|z_1 z_2| \cos \theta.$$

*Proof.* Following the hint, we let  $C = 0$ ,  $B = z_1$  and  $A = z_2$ . Then  $a = |z_1|$ ,  $b = |z_2|$  and  $c = |z_2 - z_1|$ . So

$$\begin{aligned} a^2 + b^2 - c^2 &= |z_1|^2 + |z_2|^2 - |z_2 - z_1|^2 \\ &= (z_1 \bar{z}_1 + z_2 \bar{z}_2) - (z_2 - z_1) \overline{(z_2 - z_1)} \\ &= (z_1 \bar{z}_1 + z_2 \bar{z}_2) - (z_2 - z_1)(\bar{z}_2 - \bar{z}_1) \\ &= (z_1 \bar{z}_1 + z_2 \bar{z}_2) - (z_1 \bar{z}_1 + z_2 \bar{z}_2 - z_1 \bar{z}_2 - z_2 \bar{z}_1) \\ &= z_1 \bar{z}_2 + z_2 \bar{z}_1. \end{aligned}$$

Let  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$ . Then

$$\begin{aligned} z_1 \bar{z}_2 + z_2 \bar{z}_1 &= r_1 e^{i\theta_1} \overline{r_2 e^{i\theta_2}} + r_2 e^{i\theta_2} \overline{r_1 e^{i\theta_1}} \\ &= (r_1 e^{i\theta_1})(r_2 e^{-i\theta_2}) + (r_2 e^{i\theta_2})(r_1 e^{-i\theta_1}) \\ &= r_1 r_2 e^{i(\theta_1 - \theta_2)} + r_1 r_2 e^{i(\theta_2 - \theta_1)} \\ &= 2r_1 r_2 \cos(\theta_1 - \theta_2) = 2|z_1||z_2| \cos \theta = 2ab \cos \theta. \end{aligned}$$

Therefore, we have

$$a^2 + b^2 - c^2 = z_1 \bar{z}_2 + z_2 \bar{z}_1 = 2ab \cos \theta$$

and hence

$$a^2 + b^2 - 2ab \cos \theta = c^2.$$

■

### 1.3 Exponential and Polar Form, Complex roots

1. Find the principal argument and exponential form of

(a)  $z = \frac{i}{1+i};$

(b)  $z = \sqrt{3} + i;$

(c)  $z = 2 - i.$

*Answer:*

(a)  $\text{Arg}(z) = \pi/4$  and  $z = (\sqrt{2}/2) \exp(\pi i/4)$

(b)  $\text{Arg}(z) = \pi/6$  and  $z = 2 \exp(\pi i/6)$

(c)  $\text{Arg}(z) = -\tan^{-1}(1/2)$  and  $z = \sqrt{5} \exp(-\tan^{-1}(1/2)i)$

2. Find all the complex roots of the equations

(a)  $z^6 = -9;$

(b)  $z^2 + 2z + (1 - i) = 0.$

*Solution.* (a) The roots are

$$\begin{aligned} z &= \sqrt[6]{-9} = \sqrt[6]{9e^{\pi i}} = \sqrt[3]{3} e^{\pi i/6} e^{2m\pi i/6} \quad (m = 0, 1, 2, 3, 4, 5) \\ &= \frac{3^{5/6}}{2} + \frac{\sqrt[3]{3}}{2}i, \sqrt[3]{3}i, -\frac{3^{5/6}}{2} + \frac{\sqrt[3]{3}}{2}i, -\frac{3^{5/6}}{2} - \frac{\sqrt[3]{3}}{2}i, -\sqrt[3]{3}i, \frac{3^{5/6}}{2} - \frac{\sqrt[3]{3}}{2}i. \end{aligned}$$

(b) The roots are

$$\begin{aligned}
 z &= \frac{-2 + \sqrt{4 - 4(1 - i)}}{2} = -1 + \sqrt{i} \\
 &= -1 + \sqrt{e^{\pi i/2}} = -1 + e^{\pi i/4} e^{2m\pi i/2} \quad (m = 0, 1) \\
 &= \left(-1 + \frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}i, \left(-1 - \frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}i.
 \end{aligned}$$

■

3. Find the four roots of the polynomial  $z^4 + 16$  and use these to factor  $z^4 + 16$  into two quadratic polynomials with real coefficients.

*Solution.* The four roots of  $z^4 + 16$  are given by

$$\begin{aligned}
 \sqrt[4]{-16} &= \sqrt[4]{16e^{\pi i}} = \sqrt[4]{16}e^{\pi i/4} e^{2m\pi i/4} \\
 &= 2e^{\pi i/4}, 2e^{3\pi i/4}, 2e^{5\pi i/4}, 2e^{7\pi i/4}
 \end{aligned}$$

for  $m = 0, 1, 2, 3$ . We see that these roots appear in conjugate pairs:

$$2e^{\pi i/4} = \overline{2e^{7\pi i/4}} \text{ and } 2e^{3\pi i/4} = \overline{2e^{5\pi i/4}}.$$

This gives the way to factor  $z^4 + 16$  into two quadratic polynomials of real coefficients:

$$\begin{aligned}
 z^4 + 16 &= (z - 2e^{\pi i/4})(z - 2e^{3\pi i/4})(z - 2e^{5\pi i/4})(z - 2e^{7\pi i/4}) \\
 &= \left((z - 2e^{\pi i/4})(z - 2e^{7\pi i/4})\right) \left((z - 2e^{3\pi i/4})(z - 2e^{5\pi i/4})\right) \\
 &= (z^2 - 2\operatorname{Re}(2e^{\pi i/4})z + 4)(z^2 - 2\operatorname{Re}(2e^{3\pi i/4})z + 4) \\
 &= (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4)
 \end{aligned}$$

■

4. Do the following:

(a) Use exponential form to compute

- i.  $(1 + \sqrt{3}i)^{2011}$ ;
- ii.  $(1 + \sqrt{3}i)^{-2011}$ .



(b) Prove that

$$\sum_{m=0}^{1005} \binom{2011}{2m} (-3)^m = 2^{2010}$$

and

$$\sum_{m=0}^{1005} \binom{2011}{2m+1} (-3)^m = 2^{2010}.$$

*Solution.* Since

$$1 + \sqrt{3}i = 2 \left( \frac{1}{2} + \frac{\sqrt{3}}{2}i \right) = 2 \exp \left( \frac{\pi i}{3} \right),$$

we have

$$\begin{aligned} (1 + \sqrt{3}i)^{2011} &= 2^{2011} \exp \left( \frac{2011\pi i}{3} \right) = 2^{2011} \exp \left( \frac{2011\pi i}{3} \right) \\ &= 2^{2011} \exp \left( 670\pi i + \frac{\pi i}{3} \right) \\ &= 2^{2011} \exp \left( \frac{\pi i}{3} \right) = 2^{2011} \left( \frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \\ &= 2^{2010}(1 + \sqrt{3}i). \end{aligned}$$

Similarly,

$$(1 + \sqrt{3}i)^{-2011} = 2^{-2013}(1 - \sqrt{3}i).$$

By Problem 7 in section 1.1, we have

$$\begin{aligned} 2^{2010}(1 + \sqrt{3}i) &= (1 + \sqrt{3}i)^{2011} \\ &= \sum_{m=0}^{1005} \binom{2011}{2m} (-3)^m + i \sum_{m=0}^{1005} \binom{2011}{2m+1} (-3)^m \sqrt{3}. \end{aligned}$$

It follows that

$$\sum_{m=0}^{1005} \binom{2011}{2m} (-3)^m = \sum_{m=0}^{1005} \binom{2011}{2m+1} (-3)^m = 2^{2010}.$$



5. Establish the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \quad (z \neq 1)$$

and then use it to derive **Lagrange's trigonometric identity**:

$$1 + \cos \theta + \cos 2\theta \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin \frac{(2n+1)\theta}{2}}{2 \sin \frac{\theta}{2}} \quad (0 < \theta < 2\pi).$$

**Hint:** As for the first identity, write  $S = 1 + z + z^2 + \cdots + z^n$  and consider the difference  $S - zS$ . To derive the second identity, write  $z = e^{i\theta}$  in the first one.

*Proof.* If  $z \neq 1$ , then

$$\begin{aligned} (1 - z)(1 + z + \cdots + z^n) &= 1 + z + \cdots + z^n - (z + z^2 + \cdots + z^{n+1}) \\ &= 1 - z^{n+1} \end{aligned}$$

Thus

$$1 + z + z^2 + \cdots + z^n = \begin{cases} \frac{1 - z^{n+1}}{1 - z}, & \text{if } z \neq 1 \\ n + 1, & \text{if } z = 1. \end{cases}$$

Taking  $z = e^{i\theta}$ , where  $0 < \theta < 2\pi$ , then  $z \neq 1$ . Thus

$$\begin{aligned} 1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} &= \frac{1 - e^{(n+1)\theta}}{1 - e^{i\theta}} = \frac{1 - e^{(n+1)\theta}}{-e^{i\theta/2} (e^{i\theta/2} - e^{-i\theta/2})} \\ &= \frac{-e^{-i\theta/2} (1 - e^{(n+1)\theta})}{2i \sin(\theta/2)} \\ &= \frac{i (e^{-i\theta/2} - e^{(n+1/2)\theta})}{2 \sin(\theta/2)} \\ &= \frac{1}{2} + \frac{\sin[(n + \frac{1}{2})\theta]}{2 \sin(\theta/2)} + i \frac{\cos(\theta/2) - \cos[(n + \frac{1}{2})\theta]}{2 \sin(\theta/2)} \end{aligned}$$

Equating real and imaginary parts, we obtain

$$1 + \cos \theta + \cos 2\theta \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(n + \frac{1}{2})\theta]}{2 \sin(\theta/2)}$$

and

$$\sin \theta + \sin 2\theta \cdots + \sin n\theta = \frac{\cos(\theta/2) - \cos[(n + \frac{1}{2})\theta]}{2 \sin(\theta/2)}.$$

■



# Chapter 2

## Functions

### 2.1 Basic notions and limits

1. Write the following functions  $f(z)$  in the forms  $f(z) = u(x, y) + iv(x, y)$  under Cartesian coordinates with  $u(x, y) = \operatorname{Re}(f(z))$  and  $v(x, y) = \operatorname{Im}(f(z))$ :

(a)  $f(z) = z^3 + z + 1$

(b)  $f(z) = z^3 - z$ ;

(c)  $f(z) = \frac{1}{i - z}$ ;

(d)  $f(z) = \overline{\exp(z^2)}$ .

*Solution.* (a)

$$\begin{aligned} f(z) &= (x + iy)^3 + (x + iy) + 1 \\ &= (x + iy)(x^2 - y^2 + 2ixy) + x + iy + 1 \\ &= x^3 - xy^2 + 2ix^2y + ix^2y - iy^3 - 2xy^2 + x + iy + 1 \\ &= x^3 - 3xy^2 + x + 1 + i(3x^2y - y^3 + y). \end{aligned}$$

(b)

$$\begin{aligned} f(z) &= z^3 - z = (x + yi)^3 - (x + yi) \\ &= (x^3 + 3x^2yi - 3xy^2 - y^3i) - (x + yi) \\ &= (x^3 - 3xy^2 - x) + i(3x^2y - y^3 - y), \end{aligned}$$

(c)

$$\begin{aligned}
 f(z) &= \frac{1}{i-z} = \frac{1}{-x+(1-y)i} \\
 &= \frac{-x-(1-y)i}{x^2+(1-y)^2} \\
 &= -\frac{x}{x^2+(1-y)^2} - i\frac{1-y}{x^2+(1-y)^2}
 \end{aligned}$$

(d)

$$\begin{aligned}
 f(z) &= \overline{\exp(z^2)} = \overline{\exp((x+yi)^2)} \\
 &= \overline{\exp((x^2-y^2)+2xyi)} \\
 &= e^{x^2-y^2}(\cos(2xy) + i\sin(2xy)) \\
 &= e^{x^2-y^2} \cos(2xy) - ie^{x^2-y^2} \sin(2xy)
 \end{aligned}$$

■

2. Suppose that  $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$ , where  $z = x + iy$ . Use the expressions

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}$$

to write  $f(z)$  in terms of  $z$  and simplify the result.

*Solution.* We have

$$\begin{aligned}
 f(z) &= x^2 - y^2 - 2y + i(2x - 2xy) \\
 &= x^2 - y^2 + i2x - i2xy - 2y \\
 &= (x - iy)^2 + i(2x + 2iy) \\
 &= \bar{z}^2 + 2iz.
 \end{aligned}$$

■

3. Suppose  $p(z)$  is a polynomial with real coefficients. Prove that

- (a)  $\overline{p(z)} = p(\bar{z})$ ;
- (b)  $p(z) = 0$  if and only if  $p(\bar{z}) = 0$ ;
- (c) the roots of  $p(z) = 0$  appear in conjugate pairs, i.e., if  $z_0$  is a root of  $p(z) = 0$ , so is  $\bar{z}_0$ .

*Proof.* Let  $p(z) = a_0 + a_1z + \dots + a_nz^n$  for  $a_0, a_1, \dots, a_n \in \mathbb{R}$ . Then

$$\begin{aligned}\overline{p(z)} &= \overline{a_0 + a_1z + \dots + a_nz^n} \\ &= \overline{a_0} + \overline{a_1z} + \dots + \overline{a_nz^n} \\ &= \overline{a_0} + (\overline{a_1})\overline{z} + \dots + (\overline{a_n})\overline{z}^n \\ &= a_0 + a_1\overline{z} + \dots + a_n\overline{z}^n = p(\overline{z}).\end{aligned}$$

If  $p(z) = 0$ , then  $\overline{p(z)} = 0$  and hence  $p(\overline{z}) = \overline{p(z)} = 0$ ; on the other hand, if  $p(\overline{z}) = 0$ , then  $\overline{p(\overline{z})} = p(z) = 0$  and hence  $p(z) = 0$ .

By the above,  $p(z_0) = 0$  if and only if  $p(\overline{z_0}) = 0$ . Therefore,  $z_0$  is a root of  $p(z) = 0$  if and only if  $\overline{z_0}$  is. ■

4. Compute the following limits if they exist:

(a)  $\lim_{z \rightarrow -i} \frac{iz^3 + 1}{z^2 + 1};$

(b)  $\lim_{z \rightarrow \infty} \frac{4 + z^2}{(z - 1)^2}.$

(c)  $\lim_{z \rightarrow 0} \frac{\operatorname{Im}(z)}{z}.$

*Solution.* (a)

$$\begin{aligned}\lim_{z \rightarrow -i} \frac{iz^3 + 1}{z^2 + 1} &= \lim_{z \rightarrow -i} \frac{i(z^3 + i^3)}{z^2 + 1} \\ &= \lim_{z \rightarrow -i} \frac{i(z + i)(z^2 - iz + i^2)}{(z + i)(z - i)} \\ &= \lim_{z \rightarrow -i} \frac{i(z^2 - iz + i^2)}{z - i} \\ &= i \frac{\lim_{z \rightarrow -i} (z^2 - iz + i^2)}{\lim_{z \rightarrow -i} (z - i)} \\ &= i \frac{\lim_{z \rightarrow -i} z^2 - i \lim_{z \rightarrow -i} z + \lim_{z \rightarrow -i} i^2}{\lim_{z \rightarrow -i} z - \lim_{z \rightarrow -i} i} \\ &= \frac{i((-i)^2 - i(-i) + i^2)}{-i - i} = \frac{3}{2}\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{z \rightarrow \infty} \frac{4 + z^2}{(z - 1)^2} &= \lim_{z \rightarrow 0} \frac{4 + z^{-2}}{(z^{-1} - 1)^2} \\
&= \lim_{z \rightarrow 0} \frac{4z^2 + 1}{(1 - z)^2} = \frac{\lim_{z \rightarrow 0} (4z^2 + 1)}{\lim_{z \rightarrow 0} (1 - z)^2} \\
&= \frac{4(\lim_{z \rightarrow 0} z)^2 + \lim_{z \rightarrow 0} 1}{(\lim_{z \rightarrow 0} 1 - \lim_{z \rightarrow 0} z)^2} = 1
\end{aligned}$$

(c) Since

$$\lim_{\operatorname{Re}(z)=0, z \rightarrow 0} \frac{\operatorname{Im}(z)}{z} = \lim_{y \rightarrow 0} \frac{y}{yi} = -i$$

and

$$\lim_{\operatorname{Im}(z)=0, z \rightarrow 0} \frac{\operatorname{Im}(z)}{z} = \lim_{x \rightarrow 0} \frac{0}{x} = 0$$

the limit does not exist.

5. Let

$$T(z) = \frac{z}{z + 1}.$$

Find the inverse image of the disk  $|z| < 1/2$  under  $T$  and sketch it.*Solution.* Let  $D = \{|z| < 1/2\}$ . The inverse image of  $D$  under  $T$  is

$$\begin{aligned}
T^{-1}(D) &= \{z \in \mathbb{C} : T(z) \in D\} = \{|T(z)| < \tfrac{1}{2}\} \\
&= \left\{z : \left| \frac{z}{z + 1} \right| < \tfrac{1}{2} \right\} = \{2|z| < |z + 1|\}.
\end{aligned}$$

Let  $z = x + yi$ . Then

$$\begin{aligned}
2|z| < |z + 1| &\Leftrightarrow 4(x^2 + y^2) < (x + 1)^2 + y^2 \\
&\Leftrightarrow 3x^2 - 2x + 3y^2 < 1 \\
&\Leftrightarrow \left(x - \frac{1}{3}\right)^2 + y^2 < \frac{4}{9} \\
&\Leftrightarrow \left|z - \frac{1}{3}\right| < \frac{2}{3}
\end{aligned}$$

So

$$T^{-1}(D) = \left\{z : \left|z - \frac{1}{3}\right| < \frac{2}{3}\right\}$$

is the disk with centre at  $1/3$  and radius  $2/3$ . ■

6. Sketch the following sets in the complex plane  $\mathbb{C}$  and determine whether they are open, closed, or neither; bounded; connected. Briefly state your reason.

- (a)  $|z + 3| < 1$ ;
- (b)  $|\operatorname{Im}(z)| \geq 1$ ;
- (c)  $1 \leq |z + 3| < 2$ .

*Solution.* (a) Since  $\{|z + 3| < 1\} = \{(x + 3)^2 + y^2 - 1 < 0\}$  and  $f(x, y) = (x + 3)^2 + y^2 - 1$  is a continuous function on  $\mathbb{R}^2$ , the set is open. It is not closed since the only sets that are both open and closed in  $\mathbb{C}$  are  $\emptyset$  and  $\mathbb{C}$ .

Since

$$|z| = |z + 3 - 3| \leq |z + 3| + |-3| = |z + 3| + 3 < 4$$

for all  $|z + 3| < 1$ ,  $\{|z + 3| < 1\} \subset \{|z| < 4\}$  and hence it is bounded.

It is connected since it is a convex set. ■

*Solution.* (b) We have

$$\{|\operatorname{Im}(z)| \geq 1\} = \{|y| \geq 1\} = \{y \geq 1\} \cup \{y \leq -1\}.$$

Since  $f(x, y) = y$  is continuous on  $\mathbb{R}^2$ , both  $\{y \geq 1\}$  and  $\{y \leq -1\}$  are closed and hence  $\{|\operatorname{Im}(z)| \geq 1\}$  is closed. It is not open since the only sets that are both open and closed in  $\mathbb{C}$  are  $\emptyset$  and  $\mathbb{C}$ .

Since  $z_n = n + 2i \in \{|\operatorname{Im}(z)| \geq 1\}$  for all  $n \in \mathbb{Z}$  and

$$\lim_{n \rightarrow \infty} |z_n| = \lim_{n \rightarrow \infty} \sqrt{n^2 + 4} = \infty,$$

the set is unbounded.

The set is not connected. Otherwise, let  $p = 2i$  and  $q = -2i$ . There is a polygonal path

$$\overline{p_0 p_1} \cup \overline{p_1 p_2} \cup \dots \cup \overline{p_{n-1} p_n}$$

with  $p_0 = p$ ,  $p_n = q$  and  $p_k \in \{|\operatorname{Im}(z)| \geq 1\}$  for all  $0 \leq k \leq n$ .

Let  $0 \leq m \leq n$  be the largest integer such that  $p_m \in \{y \geq 1\}$ . Then  $p_{m+1} \in \{y \leq -1\}$ . So  $\operatorname{Im}(p_m) \geq 1 > 0$  and  $\operatorname{Im}(p_{m+1}) \leq -1 < 0$ . It follows that there is a point  $p \in \overline{p_m p_{m+1}}$  such that  $\operatorname{Im}(p) = 0$ . This is a contradiction since  $\overline{p_m p_{m+1}} \subset \{|\operatorname{Im}(z)| \geq 1\}$  but  $p \notin \{|\operatorname{Im}(z)| \geq 1\}$ . Therefore the set is not connected. ■



*Solution.* (c) Since  $-2 \in \{1 \leq |z+3| < 2\}$  and  $\{|z+2| < r\} \not\subset \{1 \leq |z+3| < 2\}$  for all  $r > 0$ ,  $\{1 \leq |z+3| < 2\}$  is not open. Similarly,  $-1$  is a point lying on its complement

$$\{1 \leq |z+3| < 2\}^c = \{|z+3| \geq 2\} \cup \{|z+3| < 1\}$$

and  $\{|z+1| < r\} \not\subset \{1 \leq |z+3| < 2\}^c$  for all  $r > 0$ . Hence  $\{1 \leq |z+3| < 2\}^c$  is not open and  $\{1 \leq |z+3| < 2\}$  is not closed. In summary,  $\{1 \leq |z+3| < 2\}$  is neither open nor closed.

Since

$$|z| = |z+3-3| \leq |z+3| + |-3| < 5$$

for all  $|z+3| < 2$ ,  $\{1 \leq |z+3| < 2\} \subset \{|z| < 5\}$  and hence it is bounded.

The set is connected. To see this, we let  $p_1 = -3/2$ ,  $p_2 = -3 + 3i/2$ ,  $p_3 = -9/2$  and  $p_4 = -3 - 3i/2$ . All these points lie on the circle  $\{|z+3| = 3/2\}$  and hence lie in  $\{1 \leq |z+3| < 2\}$ .

It is easy to check that for every point  $p \in \{1 \leq |z+3| < 2\}$ ,  $\overline{pp_k} \subset \{1 \leq |z+3| < 2\}$  for at least one  $p_k \in \{p_1, p_2, p_3, p_4\}$ . So the set is connected. ■

7. Show that

$$|\sin z|^2 = (\sin x)^2 + (\sinh y)^2$$

for all complex numbers  $z = x + yi$ .

*Proof.*

$$\begin{aligned} |\sin(z)|^2 &= |\sin(x + yi)|^2 = |\sin(x) \cos(yi) + \cos(x) \sin(yi)|^2 \\ &= |\sin(x) \cosh(y) - i \cos(x) \sinh(y)|^2 \\ &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \sin^2 x (1 + \sinh^2 y) + \cos^2 x \sinh^2 y \\ &= \sin^2 x + (\cos^2 x + \sin^2 x) \sinh^2 y = (\sin x)^2 + (\sinh y)^2. \end{aligned}$$

■

8. Show that

$$|\cos(z)|^2 = (\cos x)^2 + (\sinh y)^2$$

for all  $z \in \mathbb{C}$ , where  $x = \operatorname{Re}(z)$  and  $y = \operatorname{Im}(z)$ .

*Proof.*

$$\begin{aligned}
 |\cos(z)|^2 &= |\cos(x + yi)|^2 = |\cos(x) \cos(yi) - \sin(x) \sin(yi)|^2 \\
 &= |\cos(x) \cosh(y) - i \sin(x) \sinh(y)|^2 \\
 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\
 &= \cos^2 x (1 + \sinh^2 y) + \sin^2 x \sinh^2 y \\
 &= \cos^2 x + (\cos^2 x + \sin^2 x) \sinh^2 y = (\cos x)^2 + (\sinh y)^2
 \end{aligned}$$

■

9. Show that

$$\tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - (\tan z_1)(\tan z_2)}$$

for all complex numbers  $z_1$  and  $z_2$  satisfying  $z_1, z_2, z_1 + z_2 \neq n\pi + \pi/2$  for any integer  $n$ .

*Proof.* Since

$$\begin{aligned}
 \tan z_1 + \tan z_2 &= \frac{i(e^{-iz_1} - e^{iz_1})}{e^{iz_1} + e^{-iz_1}} + \frac{i(e^{-iz_2} - e^{iz_2})}{e^{iz_2} + e^{-iz_2}} \\
 &= i \frac{(e^{-iz_1} - e^{iz_1})(e^{iz_2} + e^{-iz_2}) + (e^{-iz_2} - e^{iz_2})(e^{iz_1} + e^{-iz_1})}{(e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2})} \\
 &= -2i \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{(e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2})}
 \end{aligned}$$

and

$$\begin{aligned}
 1 - (\tan z_1)(\tan z_2) &= 1 - \left( \frac{i(e^{-iz_1} - e^{iz_1})}{e^{iz_1} + e^{-iz_1}} \right) \left( \frac{i(e^{-iz_2} - e^{iz_2})}{e^{iz_2} + e^{-iz_2}} \right) \\
 &= \frac{(e^{-iz_1} + e^{iz_1})(e^{-iz_2} + e^{iz_2}) + (e^{-iz_1} - e^{iz_1})(e^{-iz_2} - e^{iz_2})}{(e^{-iz_1} + e^{iz_1})(e^{-iz_2} + e^{iz_2})} \\
 &= 2 \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{(e^{iz_1} + e^{-iz_1})(e^{iz_2} + e^{-iz_2})},
 \end{aligned}$$

we have

$$\frac{\tan z_1 + \tan z_2}{1 - (\tan z_1)(\tan z_2)} = -i \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}} = \tan(z_1 + z_2).$$

Alternatively, we can argue as follows if we assume that the identity holds for  $z_1$  and  $z_2$  real. Let

$$F(z_1, z_2) = \tan(z_1 + z_2) - \frac{\tan z_1 + \tan z_2}{1 - (\tan z_1)(\tan z_2)}.$$

We assume that  $F(z_1, z_2) = 0$  for all  $z_1, z_2 \in \mathbb{R}$  with  $z_1, z_2, z_1 + z_2 \neq n\pi + \pi/2$ .

Fixing  $z_1 \in \mathbb{R}$ , we let  $f(z) = F(z_1, z)$ . Then  $f(z)$  is analytic in its domain

$$\mathbb{C} \setminus (\{n\pi + \pi/2\} \cup \{n\pi + \pi/2 - z_1\}).$$

And we know that  $f(z) = 0$  for  $z$  real. Therefore, by the uniqueness of analytic functions,  $f(z) \equiv 0$  in its domain. So  $F(z_1, z_2) = 0$  for all  $z_1 \in \mathbb{R}$  and  $z_2 \in \mathbb{C}$  in its domain.

Fixing  $z_2 \in \mathbb{C}$ , we let  $g(z) = F(z, z_2)$ . Then  $g(z)$  is analytic in its domain

$$\mathbb{C} \setminus (\{n\pi + \pi/2\} \cup \{n\pi + \pi/2 - z_2\}).$$

And we have proved that  $g(z) = 0$  for  $z$  real. Therefore, by the uniqueness of analytic functions,  $g(z) \equiv 0$  in its domain. Hence  $F(z_1, z_2) = 0$  for all  $z_1 \in \mathbb{C}$  and  $z_2 \in \mathbb{C}$  in its domain. ■

10. Find all the complex roots of the equation  $\cos z = 3$ .

*Solution.* Since  $\cos z = (e^{iz} + e^{-iz})/2$ , it comes down to solve the equation  $e^{iz} + e^{-iz} = 6$ , i.e.,

$$w + w^{-1} = 6 \Leftrightarrow w^2 - 6w + 1 = 0$$

if we let  $w = e^{iz}$ . The roots of  $w^2 - 6w + 1 = 0$  are  $w = 3 \pm 2\sqrt{2}$ . Therefore, the solutions for  $\cos z = 3$  are

$$iz = \log(3 \pm 2\sqrt{2}) \Leftrightarrow z = -i(\ln(3 \pm 2\sqrt{2}) + 2n\pi i) = 2n\pi - i\ln(3 \pm 2\sqrt{2})$$

for  $n$  integers. ■

11. Calculate  $\sin\left(\frac{\pi}{4} + i\right)$ .

*Solution.*

$$\begin{aligned}\sin\left(\frac{\pi}{4} + i\right) &= \frac{1}{2i}(e^{i(\pi/4+i)} - e^{-i(\pi/4+i)}) \\ &= \frac{1}{2i}(e^{-1}e^{\pi i/4} - ee^{-\pi i/4}) \\ &= \frac{1}{2i}\left(e^{-1}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) - e\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right) \\ &= \frac{\sqrt{2}}{4}\left(e + \frac{1}{e}\right) + \frac{\sqrt{2}}{4}\left(e - \frac{1}{e}\right)i\end{aligned}$$

■

12. Compute  $\cos\left(\frac{\pi}{3} + i\right)$ .

*Solution.*

$$\begin{aligned}\cos\left(\frac{\pi}{3} + i\right) &= \frac{1}{2}(e^{i(\pi/3+i)} + e^{-i(\pi/3+i)}) \\ &= \frac{1}{2}(e^{-1}e^{\pi i/3} + ee^{-\pi i/3}) \\ &= \frac{1}{2}\left(e^{-1}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) + e\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)\right) \\ &= \frac{1}{4}\left(e + \frac{1}{e}\right) - \frac{\sqrt{3}i}{4}\left(e - \frac{1}{e}\right)\end{aligned}$$

■

13. Find  $i^i$  and its principal value.

*Solution.* We have

$$i^i = e^{i \log i} = e^{i(2n\pi i + \pi i/2)} = e^{-2n\pi - \pi/2}$$

for  $n$  integers and its principal value given by

$$i^i = e^{i \operatorname{Log} i} = e^{i(\pi i/2)} = e^{-\pi/2}.$$

■

14. Let  $f(z)$  be the principal branch of  $\sqrt[3]{z}$ .

(a) Find  $f(-i)$ .

*Solution.*

$$f(-i) = \exp\left(\frac{1}{3} \operatorname{Log}(-i)\right) = \exp\left(\frac{1}{3}\left(-\frac{\pi i}{2}\right)\right) = \exp\left(-\frac{\pi i}{6}\right) = \frac{\sqrt{3}}{2} - \frac{i}{2}$$

■

(b) Show that

$$f(z_1)f(z_2) = \lambda f(z_1z_2)$$

for all  $z_1, z_2 \neq 0$ , where  $\lambda = 1, \frac{-1 + \sqrt{3}i}{2}$  or  $\frac{-1 - \sqrt{3}i}{2}$ .

*Proof.* Since

$$\begin{aligned} \frac{f(z_1)f(z_2)}{f(z_1z_2)} &= \exp\left(\frac{1}{3} \operatorname{Log} z_1 + \frac{1}{3} \operatorname{Log} z_2 - \frac{1}{3} \operatorname{Log}(z_1z_2)\right) \\ &= \exp\left(\frac{1}{3}(\operatorname{Log} z_1 + \operatorname{Log} z_2 - \operatorname{Log}(z_1z_2))\right) \\ &= \exp\left(\frac{i}{3}(\operatorname{Arg} z_1 + \operatorname{Arg} z_2 - \operatorname{Arg}(z_1z_2))\right) = \exp\left(\frac{2n\pi i}{3}\right) \end{aligned}$$

for some integer  $n$ ,  $\lambda = \exp(2n\pi i/3)$ . Therefore,

$$\lambda = \begin{cases} 1 & \text{if } n = 3k \\ \frac{-1 + \sqrt{3}i}{2} & \text{if } n = 3k + 1 \\ \frac{-1 - \sqrt{3}i}{2} & \text{if } n = 3k + 2 \end{cases}$$

where  $k \in \mathbb{Z}$ .

■

15. Let  $f(z)$  be the principal branch of  $z^{-i}$ .

(a) Find  $f(i)$ .

*Solution.*

$$f(i) = i^{-i} = \exp(-i \operatorname{Log}(i)) = \exp(-i(\pi i/2)) = e^{\pi/2}.$$

■

(b) Show that

$$f(z_1)f(z_2) = \lambda f(z_1z_2)$$

for all  $z_1, z_2 \neq 0$ , where  $\lambda = 1, e^{2\pi}$  or  $e^{-2\pi}$ .

*Proof.* Since

$$\begin{aligned} \frac{f(z_1)f(z_2)}{f(z_1z_2)} &= \exp(-i \operatorname{Log} z_1 - i \operatorname{Log} z_2 + i \operatorname{Log}(z_1z_2)) \\ &= \exp(-i(\operatorname{Log} z_1 + \operatorname{Log} z_2 - \operatorname{Log}(z_1z_2))) \\ &= \exp(-i(i \operatorname{Arg} z_1 + i \operatorname{Arg} z_2 - i \operatorname{Arg}(z_1z_2))) \\ &= \exp(\operatorname{Arg} z_1 + \operatorname{Arg} z_2 - \operatorname{Arg}(z_1z_2)) \\ &= \exp(2n\pi) \end{aligned}$$

for some integer  $n$ ,  $\lambda = \exp(2n\pi)$ . And since

$$-\pi < \operatorname{Arg}(z_1) \leq \pi, -\pi < \operatorname{Arg}(z_2) \leq \pi$$

and

$$-\pi < \operatorname{Arg}(z_1z_2) \leq \pi,$$

we conclude that

$$-3\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 - \operatorname{Arg}(z_1z_2) < 3\pi$$

and hence  $-3 < 2n < 3$ . So  $n = -1, 0$  or  $1$  and  $\lambda = e^{-2\pi}, 1$  or  $e^{2\pi}$ . ■

## 2.2 Continuity and Differentiation

1. Show that if  $f(z)$  is continuous at  $z_0$ , so is  $|f(z)|$ .

*Proof.* Let  $f(z) = u(x, y) + iv(x, y)$ . Since  $f(z)$  is continuous at  $z_0 = x_0 + y_0i$ ,  $u(x, y)$  and  $v(x, y)$  are continuous at  $(x_0, y_0)$ . Therefore,

$$(u(x, y))^2 + (v(x, y))^2$$

is continuous at  $(x_0, y_0)$  since the sums and products of continuous functions are continuous. It follows that

$$|f(z)| = \sqrt{(u(x, y))^2 + (v(x, y))^2}$$

is continuous at  $z_0$  since the compositions of continuous functions are continuous. ■

2. Let

$$f(z) = \begin{cases} \bar{z}^3/z^2 & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Show that

- (a)  $f(z)$  is continuous everywhere on  $\mathbb{C}$ ;
- (b) the complex derivative  $f'(0)$  does not exist.

*Proof.* Since both  $\bar{z}^3$  and  $z^2$  are continuous on  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  and  $z^2 \neq 0$ ,  $f(z) = \bar{z}^3/z^2$  is continuous on  $\mathbb{C}^*$ .

At  $z = 0$ , we have

$$\lim_{z \rightarrow 0} |f(z)| = \lim_{z \rightarrow 0} \left| \frac{\bar{z}^3}{z^2} \right| = \lim_{z \rightarrow 0} |z| = 0$$

and hence  $\lim_{z \rightarrow 0} f(z) = 0 = f(0)$ . So  $f$  is also continuous at 0 and hence continuous everywhere on  $\mathbb{C}$ .

The complex derivative  $f'(0)$ , if exists, is given by

$$\lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \rightarrow 0} \frac{\bar{z}^3}{z^3}.$$

Let  $z = x + yi$ . If  $y = 0$  and  $x \rightarrow 0$ , then

$$\lim_{z=x \rightarrow 0} \frac{\bar{z}^3}{z^3} = \lim_{x \rightarrow 0} \frac{x^3}{x^3} = 1.$$

On the other hand, if  $x = 0$  and  $y \rightarrow 0$ , then

$$\lim_{z=yi \rightarrow 0} \frac{\bar{z}^3}{z^3} = \lim_{y \rightarrow 0} \frac{(-yi)^3}{(yi)^3} = -1.$$

So the limit  $\lim_{z \rightarrow 0} \bar{z}^3/z^3$  and hence  $f'(0)$  do not exist. ■

3. Show that  $f(z)$  in (2) is actually nowhere differentiable, i.e., the complex derivative  $f'(z)$  does not exist for any  $z \in \mathbb{C}$ .

*Proof.* It suffices to show that C-R equations fail at every  $z \neq 0$ :

$$\begin{aligned}
 \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f(z) &= \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \bar{z}^3 \\
 &= \frac{\partial}{\partial x} \left( \frac{\bar{z}^3}{z^2} \right) + i \frac{\partial}{\partial y} \left( \frac{\bar{z}^3}{z^2} \right) \\
 &= \left( \frac{3\bar{z}^2}{z^2} - \frac{2\bar{z}^3}{z^3} \right) + i \left( -\frac{3i\bar{z}^2}{z^2} - \frac{2i\bar{z}^3}{z^3} \right) \\
 &= \frac{6\bar{z}^2}{z^2} \neq 0
 \end{aligned}$$

for  $z \neq 0$ . ■

4. Find  $f'(z)$  when

(a)  $f(z) = z^2 - 4z + 2$ ;

(b)  $f(z) = (1 - z^2)^4$ ;

(c)  $f(z) = \frac{z+1}{2z+1}$  ( $z \neq -\frac{1}{2}$ );

(d)  $f(z) = e^{1/z}$  ( $z \neq 0$ ).

*Answer.* (a)  $2z - 4$ , (b)  $-8(1 - z^2)^3 z$ , (c)  $-1/(2z + 1)^2$ , (d)  $-e^{1/z}/z^2$

5. Prove the following version of complex L'Hospital: Let  $f(z)$  and  $g(z)$  be two complex functions defined on  $|z - z_0| < r$  for some  $r > 0$ . Suppose that  $f(z_0) = g(z_0) = 0$ ,  $f(z)$  and  $g(z)$  are differentiable at  $z_0$  and  $g'(z_0) \neq 0$ . Then

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

[Refer to: problems 2c and 7 in section 3.1; and problem 9 in section 3.2]

*Proof.* Since  $f(z)$  and  $g(z)$  are differentiable at  $z_0$ , we have

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

and

$$\lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{z - z_0} = g'(z_0).$$



And since  $g'(z_0) \neq 0$ ,

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \frac{\lim_{z \rightarrow z_0} (f(z) - f(z_0)) / (z - z_0)}{\lim_{z \rightarrow z_0} (g(z) - g(z_0)) / (z - z_0)} = \frac{f'(z_0)}{g'(z_0)}.$$

Finally, since  $f(z_0) = g(z_0) = 0$ ,

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

■

6. Show that if  $f(z)$  satisfies the Cauchy-Riemann equations at  $z_0$ , so does  $(f(z))^n$  for every positive integer  $n$ .

*Proof.* Since  $f(z)$  satisfies the Cauchy-Riemann equations at  $z_0$ ,

$$\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f(z) = 0$$

at  $z_0$ . Therefore,

$$\begin{aligned} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (f(z))^n &= \frac{\partial}{\partial x} (f(z))^n + i \frac{\partial}{\partial y} (f(z))^n \\ &= (f(z))^{n-1} \frac{\partial}{\partial x} f(z) + i (f(z))^{n-1} \frac{\partial}{\partial y} f(z) \\ &= (f(z))^{n-1} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f(z) = 0 \end{aligned}$$

at  $z_0$ .

■

## 2.3 Analytic functions

1. Explain why the function  $f(z) = 2z^2 - 3 - ze^z + e^{-z}$  is entire.

*Proof.* Since every polynomial is entire,  $2z^2 - 3$  is entire; since both  $-z$  and  $e^z$  are entire, their product  $-ze^z$  is entire; since  $e^z$  and  $-z$  are entire, their composition  $e^{-z}$  is entire. Finally,  $f(z)$  is the sum of  $2z^2 - 3$ ,  $-ze^z$  and  $e^{-z}$  and hence entire.

■

2. Let  $f(z)$  be an analytic function on a connected open set  $D$ . If there are two constants  $c_1$  and  $c_2 \in \mathbb{C}$ , not all zero, such that  $c_1 f(z) + c_2 \overline{f(z)} = 0$  for all  $z \in D$ , then  $f(z)$  is a constant on  $D$ .

*Proof.* If  $c_2 = 0$ ,  $c_1 \neq 0$  since  $c_1$  and  $c_2$  cannot be both zero. Then we have  $c_1 f(z) = 0$  and hence  $f(z) = 0$  for all  $z \in D$ .

If  $c_2 \neq 0$ ,  $\overline{f(z)} = -(c_1/c_2)f(z)$ . And since  $f(z)$  is analytic in  $D$ ,  $\overline{f(z)}$  is analytic in  $D$ . So both  $f(z)$  and  $\overline{f(z)}$  are analytic in  $D$ . Therefore, both  $f(z)$  and  $\overline{f(z)}$  satisfy Cauchy-Riemann equations in  $D$ . Hence

$$\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u + vi) = 0$$

and

$$\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u - vi) = 0$$

in  $D$ , where  $f(z) = u(x, y) + iv(x, y)$  with  $u = \operatorname{Re}(f)$  and  $v = \operatorname{Im}(f)$ . It follows that

$$\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) v = 0$$

and hence  $u_x = u_y = v_x = v_y = 0$  in  $D$ . Therefore,  $u$  and  $v$  are constants on  $D$  and hence  $f(z) \equiv \text{const.}$  ■

3. Show that the function  $\sin(\bar{z})$  is nowhere analytic on  $\mathbb{C}$ .

*Proof.* Since

$$\begin{aligned} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \sin(\bar{z}) &= \frac{\partial}{\partial x} \sin(\bar{z}) + i \frac{\partial}{\partial y} \sin(\bar{z}) \\ &= \cos(\bar{z}) \frac{\partial \bar{z}}{\partial x} + i \cos(\bar{z}) \frac{\partial \bar{z}}{\partial y} \\ &= \cos(\bar{z}) + i \cos(\bar{z})(-i) = 2 \cos(\bar{z}) \end{aligned}$$

$\sin(\bar{z})$  is not differentiable and hence not analytic at every point  $z$  satisfying  $\cos(\bar{z}) \neq 0$ . At every point  $z_0$  satisfying  $\cos(\bar{z}_0) = 0$ , i.e.,  $z_0 = n\pi + \pi/2$ ,  $\sin(\bar{z})$  is not differentiable in  $|z - z_0| < r$  for all  $r > 0$ . Hence  $\sin(\bar{z})$  is not analytic at  $z_0 = n\pi + \pi/2$  either. In conclusion,  $\sin(\bar{z})$  is nowhere analytic. ■

4. Let  $f(z) = u(x, y) + iv(x, y)$  be an entire function satisfying  $u(x, y) \leq x$  for all  $z = x + yi$ . Show that  $f(z)$  is a polynomial of degree at most one.

*Proof.* Let  $g(z) = \exp(f(z) - z)$ . Then  $|g(z)| = \exp(u(x, y) - x)$ . Since  $u(x, y) \leq x$ ,  $|g(z)| \leq 1$  for all  $z$ . And since  $g(z)$  is entire,  $g(z)$  must be constant by Liouville's theorem. Therefore,  $g'(z) \equiv 0$ . That is,  $(f'(z) - 1) \exp(f(z) - z) \equiv 0$  and hence  $f'(z) = 1$  for all  $z$ . So  $f(z) \equiv z + c$  for some constant  $c$ . ■

5. Show that

$$|\exp(z^3 + i) + \exp(-iz^2)| \leq e^{x^3 - 3xy^2} + e^{2xy}$$

where  $x = \operatorname{Re}(z)$  and  $y = \operatorname{Im}(z)$ .

*Proof.* Note that  $|e^z| = e^{\operatorname{Re}(z)}$ . Therefore,

$$\begin{aligned} |\exp(z^3 + i) + \exp(-iz^2)| &\leq |\exp(z^3 + i)| + |\exp(-iz^2)| \\ &= \exp(\operatorname{Re}(z^3 + i)) + \exp(\operatorname{Re}(-iz^2)) \\ &= \exp(\operatorname{Re}((x^3 - 3xy^2) + (3x^2y - y^3 + 1)i)) \\ &\quad + \exp(\operatorname{Re}(2xy - (x^2 - y^2)i)) \\ &= e^{x^3 - 3xy^2} + e^{2xy}. \end{aligned}$$

■

6. Let  $f(z) = u(x, y) + iv(x, y)$  be an entire function satisfying

$$v(x, y) \geq x$$

for all  $z = x + yi$ , where  $u(x, y) = \operatorname{Re}(f(z))$  and  $v(x, y) = \operatorname{Im}(f(z))$ . Show that  $f(z)$  is a polynomial of degree 1.

*Proof.* Let  $g(z) = \exp(if(z) + z)$ . Then

$$|g(z)| = |\exp((-v(x, y) + iu(x, y)) + (x + iy))| = \exp(x - v(x, y)).$$

Since  $v(x, y) \geq x$ ,  $x - v(x, y) \leq 0$  and  $|g(z)| \leq 1$  for all  $z$ . And since  $g(z)$  is entire,  $g(z)$  must be constant by Liouville's theorem. Therefore,  $g'(z) \equiv 0$ . That is,  $(if'(z) + 1) \exp(if(z) + z) \equiv 0$  and hence  $f'(z) = i$  for all  $z$ . So  $f(z) \equiv iz + c$  for some constant  $c$ . ■

7. Show that the entire function  $\cosh(z)$  takes every value in  $\mathbb{C}$  infinitely many times.

*Proof.* For every  $w_0 \in \mathbb{C}$ , the quadratic equation  $y^2 - 2w_0y + 1 = 0$  has a complex root  $y_0$ . We cannot have  $y_0 = 0$  since  $0^2 - 2w_0 \cdot 0 + 1 \neq 0$ . Therefore,  $y_0 \neq 0$  and there is  $z_0 \in \mathbb{C}$  such that  $e^{z_0} = y_0$ . Then

$$\cosh(z_0) = \frac{e^{z_0} + e^{-z_0}}{2} = \frac{y_0^2 + 1}{2y_0} = \frac{2w_0y_0}{2y_0} = w_0.$$

And since  $\cosh(z + 2\pi i) = \cosh(z)$ ,  $\cosh(z_0 + 2n\pi i) = w_0$  for all integers  $n$ . Therefore,  $\cosh(z)$  takes every value  $w_0$  infinitely many times. ■

8. Determine which of the following functions  $f(z)$  are entire and which are not? You must justify your answer. Also find the complex derivative  $f'(z)$  of  $f(z)$  if  $f(z)$  is entire. Here  $z = x + yi$  with  $x = \operatorname{Re}(z)$  and  $y = \operatorname{Im}(z)$ .

(a)  $f(z) = \frac{1}{1 + |z|^2}$

*Solution.* Since  $u(x, y) = \operatorname{Re}(f(z)) = (1 + x^2 + y^2)^{-1}$  and  $v(x, y) = 0$ ,  $u_x = 2x(1 + x^2 + y^2)^{-2} \neq 0 = v_y$ . Hence the Cauchy-Riemann equations fail for  $f(z)$  and  $f(z)$  is not entire.

(b)  $f(z) = 2^{(3^z)}$  (here  $2^z$  and  $3^z$  are taken to be the principle values of  $2^z$  and  $3^z$ , respectively, by convention)

*Solution.* Let  $g(z) = 2^z$  and  $h(z) = 2^{3^z}$ . Since both  $g(z)$  and  $h(z)$  are entire,  $f(z) = g(h(z))$  is entire and

$$f'(z) = g'(h(z))h'(z) = 2^{3^z} 3^z (\ln 2)(\ln 3)$$

by chain rule.

(c)  $f(z) = (x^2 - y^2) - 2xyi$

*Solution.* Since

$$\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f = (2x - 2yi) + i(-2y - 2xi) = 4x - 4yi \neq 0,$$

the Cauchy-Riemann equations fail for  $f(z)$  and hence  $f(z)$  is not entire.

(d)  $f(z) = (x^2 - y^2) + 2xyi$

*Solution.* Since

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f = (2x - 2yi) + i(2y + 2xi) = 0,$$

the Cauchy-Riemann equations hold for  $f(z)$  everywhere. And since  $f_x$  and  $f_y$  are continuous,  $f(z)$  is analytic on  $\mathbb{C}$ . And  $f'(z) = f_x = 2x + 2yi = 2z$ .

9. Let  $C_R$  denote the upper half of the circle  $|z| = R$  for some  $R > 1$ . Show that

$$\left|\frac{e^{iz}}{z^2 + z + 1}\right| \leq \frac{1}{(R-1)^2}$$

for all  $z$  lying on  $C_R$ .

*Proof.* For  $z \in C_R$ ,  $|z| = R$  and  $\text{Im}(z) \geq 0$ . Let  $z = x + yi$ . Since  $y = \text{Im}(z) \geq 0$ ,

$$|e^{iz}| = |e^{i(x+yi)}| = |e^{-y+xi}| = e^{-y} \leq 1$$

for  $z \in C_R$ . And since

$$\begin{aligned} |z^2 + z + 1| &= \left| \left(z - \frac{-1 + \sqrt{3}i}{2}\right) \left(z - \frac{-1 - \sqrt{3}i}{2}\right) \right| \\ &= \left| z - \frac{-1 + \sqrt{3}i}{2} \right| \left| z - \frac{-1 - \sqrt{3}i}{2} \right| \\ &\geq \left( |z| - \left| \frac{-1 + \sqrt{3}i}{2} \right| \right) \left( |z| - \left| \frac{-1 - \sqrt{3}i}{2} \right| \right) \\ &= (R-1)(R-1) = (R-1)^2, \end{aligned}$$

we obtain

$$\left|\frac{e^{iz}}{z^2 + z + 1}\right| \leq \frac{1}{(R-1)^2}$$

for  $z \in C_R$ . ■

10. Let

$$f(z) = \begin{cases} \bar{z}^2 / |z| & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Show that  $f(z)$  is continuous everywhere but nowhere analytic on  $\mathbb{C}$ .

*Proof.* Since both  $\bar{z}$  and  $|z|$  are continuous on  $\mathbb{C}$ ,  $\bar{z}^2/|z|$  is continuous on  $\mathbb{C}^*$ . Therefore,  $f(z)$  is continuous on  $\mathbb{C}^*$ . To see that it is continuous at 0, we just have to show that

$$\lim_{z \rightarrow 0} f(z) = \lim_{z \rightarrow 0} \frac{\bar{z}^2}{|z|} = f(0) = 0.$$

This follows from

$$\lim_{z \rightarrow 0} \left| \frac{\bar{z}^2}{|z|} \right| = \lim_{z \rightarrow 0} \frac{|\bar{z}|^2}{|z|} = \lim_{z \rightarrow 0} |z| = 0.$$

Therefore,  $f(z)$  is continuous everywhere on  $\mathbb{C}$ .

To show that  $f(z)$  is nowhere analytic, it suffices to show that the Cauchy-Riemann equations fail for  $f(z)$  on  $\mathbb{C}^*$ . This follows from

$$\begin{aligned} & \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \left( \frac{\bar{z}^2}{|z|} \right) \\ &= \left( \frac{2\bar{z}}{|z|} - \frac{x\bar{z}^2}{|z|^3} \right) + i \left( -\frac{2i\bar{z}}{|z|} - \frac{y\bar{z}^2}{|z|^3} \right) \\ &= \frac{(4z - x - iy)\bar{z}^2}{|z|^3} = \frac{3\bar{z}}{|z|} \neq 0 \end{aligned}$$

for  $z \neq 0$ . Consequently,  $f(z)$  is nowhere analytic. ■

11. Find where

$$\tan^{-1}(z) = \frac{i}{2} \operatorname{Log} \frac{i+z}{i-z}$$

is analytic?

*Solution.* The branch locus of  $\tan^{-1}(z)$  is

$$\left\{ z : \frac{i+z}{i-z} = w \in (-\infty, 0] \right\} = \left\{ z : z = i \frac{w-1}{w+1}, w \in (-\infty, 0] \right\}.$$

For  $w \in (-\infty, 0]$ ,

$$\frac{w-1}{w+1} = 1 - \frac{2}{w+1} \in (-\infty, -1] \cup (1, \infty)$$

so  $\tan^{-1}(z)$  is analytic in

$$\mathbb{C} \setminus \{z : \operatorname{Re}(z) = 0, \operatorname{Im}(z) \in (-\infty, -1] \cup [1, \infty)\}.$$



# Chapter 3

## Complex Integrals

### 3.1 Contour integrals

1. Show that

$$|\operatorname{Log}(z)| \leq |\ln |z|| + \pi \quad (3.1.1)$$

for all  $z \neq 0$ .

*Proof.* Since  $\operatorname{Log}(z) = \ln |z| + i \operatorname{Arg}(z)$  for  $-\pi < \operatorname{Arg}(z) \leq \pi$ ,

$$|\operatorname{Log}(z)| = |\ln |z| + i \operatorname{Arg}(z)| \leq |\ln |z|| + |i \operatorname{Arg}(z)| \leq |\ln |z|| + \pi.$$

■

2. Evaluate the following integrals:

(a)  $\int_1^2 (t^2 + i)^2 dt;$

(b)  $\int_0^{\pi/4} e^{-2it} dt;$

(c)  $\int_0^\infty t e^{zt} dt$  when  $\operatorname{Re}(z) < 0$ .



*Solution.*

(a)

$$\begin{aligned}\int_1^2 (t^2 + i)^2 dt &= \int_1^2 (t^4 + 2it^2 + i^2) dt \\ &= \frac{t^5}{5} + \frac{2it^3}{3} - t \Big|_1^2 = \frac{26}{5} + \frac{14}{3}i\end{aligned}$$

(b)

$$\begin{aligned}\int_0^{\pi/4} e^{-2it} dt &= -\frac{e^{-2it}}{2i} \Big|_0^{\pi/4} \\ &= \frac{1+i}{2i} = \frac{1}{2} - \frac{i}{2}\end{aligned}$$

(c)

$$\begin{aligned}\int_0^\infty te^{zt} dt &= \frac{1}{z} \int_0^\infty t d(e^{zt}) \\ &= \frac{1}{z} \left( te^{zt} \Big|_0^\infty - \int_0^\infty e^{zt} dt \right) \\ &= \frac{1}{z} \left( \lim_{t \rightarrow \infty} te^{zt} - \frac{1}{z} \left( \lim_{t \rightarrow \infty} e^{zt} - 1 \right) \right) \\ &= \frac{1}{z^2}\end{aligned}$$

where

$$\lim_{t \rightarrow \infty} te^{zt} = \lim_{t \rightarrow \infty} e^{zt} = 0$$

because

$$\lim_{t \rightarrow \infty} |te^{zt}| = \lim_{t \rightarrow \infty} te^{t \operatorname{Re}(z)} = \lim_{t \rightarrow \infty} \frac{t}{e^{-xt}} = -\lim_{t \rightarrow \infty} \frac{1}{xe^{-xt}} = 0$$

by L'Hospital (see Problem 5 in section 2.2), and

$$\lim_{t \rightarrow \infty} |e^{zt}| = \lim_{t \rightarrow \infty} e^{t \operatorname{Re}(z)} = \lim_{t \rightarrow \infty} \frac{1}{e^{-xt}} = 0$$

as  $x = \operatorname{Re}(z) < 0$ .

3. Find the contour integral  $\int_\gamma \bar{z} dz$  for

- (a)  $\gamma$  is the triangle  $ABC$  oriented counterclockwise, where  $A = 0$ ,  $B = 1 + i$  and  $C = -2$ ;  
 (b)  $\gamma$  is the circle  $|z - i| = 2$  oriented counterclockwise.

*Solution.* (a)

$$\begin{aligned}
 \int_{\gamma} \bar{z} dz &= \int_{AB} \bar{z} dz + \int_{BC} \bar{z} dz + \int_{CA} \bar{z} dz \\
 &= \int_0^1 \overline{t(1+i)} d(t(1+i)) \\
 &\quad + \int_0^1 \overline{(1-t)(1+i) - 2t} d((1-t)(1+i) - 2t) \\
 &\quad + \int_0^1 \overline{-2(1-t)} d(-2(1-t)) \\
 &= \int_0^1 2t dt + \int_0^1 ((2i-4) + 10t) dt + \int_0^1 4(t-1) dt \\
 &= 1 + (2i-4) + 5 - 2 = 2i
 \end{aligned}$$

(b)

$$\int_0^{2\pi} \overline{i + 2e^{it}} d(i + 2e^{it}) = \int_0^{2\pi} 2i(-i + 2e^{-it})e^{it} dt = 8\pi i.$$

■

4. Compute the following contour integral

$$\int_L \bar{z} dz,$$

where  $L$  is the boundary of the triangle  $ABC$  with  $A = 0$ ,  $B = 1$  and  $C = i$ , oriented counter-clockwise.

*Solution.*

$$\begin{aligned}
 \int_L \bar{z} dz &= \int_{AB} \bar{z} dz + \int_{BC} \bar{z} dz + \int_{CA} \bar{z} dz \\
 &= \int_0^1 \bar{t} dt + \int_0^1 \overline{(1-t) + ti} d((1-t) + ti) \\
 &\quad + \int_0^1 \overline{(1-t)i} d((1-t)i) \\
 &= \int_0^1 t dt + (-1+i) \int_0^1 ((1-t) - ti) dt - \int_0^1 (1-t) dt = i
 \end{aligned}$$

■

5. Evaluate the contour integral

$$\int_C f(z) dz$$

using the parametric representations for  $C$ , where

$$f(z) = \frac{z^2 - 1}{z}$$

and the curve  $C$  is

- (a) the semicircle  $z = 2e^{i\theta}$  ( $0 \leq \theta \leq \pi$ );
- (b) the semicircle  $z = 2e^{i\theta}$  ( $\pi \leq \theta \leq 2\pi$ );
- (c) the circle  $z = 2e^{i\theta}$  ( $0 \leq \theta \leq 2\pi$ ).

*Solution.*

(a)

$$\int_C f(z) dz = \int_0^\pi \frac{4e^{2i\theta} - 1}{2e^{i\theta}} d(2e^{i\theta}) = (2e^{2i\theta} - i) \Big|_0^\pi = -\pi i$$

(b)

$$\int_C f(z) dz = \int_\pi^{2\pi} \frac{4e^{2i\theta} - 1}{2e^{i\theta}} d(2e^{i\theta}) = (2e^{2i\theta} - i) \Big|_\pi^{2\pi} = -\pi i$$

(c) Summing (a) and (b), we have  $-2\pi i$ .

6. Redo previous Problem 5 using an antiderivative of  $f(z)$ .

*Solution.* For (a),

$$\begin{aligned} \int_C f(z) dz &= \frac{z^2}{2} \Big|_2^{-2} - \left( \lim_{\substack{z \rightarrow -2 \\ \text{Im}(z) > 0}} \text{Log}(z) - \text{Log}(2) \right) \\ &= -(\ln 2 + \pi i - \ln 2) = -\pi i. \end{aligned}$$

For (b),

$$\begin{aligned} \int_C f(z) dz &= \frac{z^2}{2} \Big|_{-2}^2 - \left( \text{Log}(2) - \lim_{\substack{z \rightarrow -2 \\ \text{Im}(z) < 0}} \text{Log}(z) \right) \\ &= -(\ln 2 - (\ln 2 - \pi i)) = -\pi i. \end{aligned}$$

7. Let  $C_R$  be the circle  $|z| = R$  ( $R > 1$ ) oriented counterclockwise. Show that

$$\left| \int_{C_R} \frac{\text{Log}(z^2)}{z^2} dz \right| < 4\pi \left( \frac{\pi + \ln R}{R} \right)$$

and then

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{\text{Log}(z^2)}{z^2} dz = 0.$$

*Proof.* Using expression (3.1.1) in Problem 1, we have

$$|\text{Log}(z^2)| \leq |\ln |z^2|| + \pi = 2 \ln R + \pi$$

for  $|z| = R > 1$ . Therefore,

$$\begin{aligned} \left| \int_{C_R} \frac{\text{Log}(z^2)}{z^2} dz \right| &\leq 2\pi R \left( \frac{\pi + 2 \ln R}{R^2} \right) \\ &= 4\pi \left( \frac{\pi/2 + \ln R}{R} \right) < 4\pi \left( \frac{\pi + \ln R}{R} \right). \end{aligned}$$

And since

$$\lim_{R \rightarrow \infty} 4\pi \left( \frac{\pi + \ln R}{R} \right) = 4\pi \lim_{R \rightarrow \infty} \frac{1}{R} = 0$$

by L'Hospital (see Problem 5 in section 2.2),

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{\text{Log}(z^2)}{z^2} dz = 0.$$

■

8. Without evaluating the integral, show that

$$\left| \int_C \frac{dz}{\bar{z}^2 + \bar{z} + 1} \right| \leq \frac{9\pi}{16}$$

where  $C$  is the arc of the circle  $|z| = 3$  from  $z = 3$  to  $z = 3i$  lying in the first quadrant.

*Proof.* Since

$$|\bar{z}^2 + \bar{z} + 1| \geq |\bar{z}^2| - |\bar{z}| - 1 = |z|^2 - |z| - 1 = 5$$

for  $|z| = 3$ ,

$$\left| \frac{1}{\bar{z}^2 + \bar{z} + 1} \right| \leq \frac{1}{5}.$$

Therefore,

$$\left| \int_C \frac{dz}{\bar{z}^2 + \bar{z} + 1} \right| \leq \frac{6\pi}{4} \left( \frac{1}{5} \right) = \frac{3\pi}{10} < \frac{9\pi}{16}.$$

■

### 3.2 Cauchy Integral Theorem and Cauchy Integral Formula

1. Let  $C$  be the boundary of the triangle with vertices at the points  $0$ ,  $3i$  and  $-4$  oriented counterclockwise. Compute the contour integral

$$\int_C (e^z - \bar{z}) dz.$$

*Solution.* By Cauchy Integral Theorem,  $\int_C e^z dz = 0$  since  $C$  is closed and  $e^z$  is entire. Therefore,

$$\begin{aligned} \int_C (e^z - \bar{z}) dz &= - \int_C \bar{z} dz = - \int_{p_1 p_2} \bar{z} dz - \int_{p_2 p_3} \bar{z} dz - \int_{p_3 p_1} \bar{z} dz \\ &= - \int_0^1 (-3it) d(3it) - \int_0^1 (-3i(1-t) - 4t) d(3i(1-t) - 4t) \\ &\quad - \int_0^1 (-4)(1-t) d((-4)(1-t)) \\ &= -\frac{9}{2} - \frac{7}{2} - 12i + 8 = -12i \end{aligned}$$

where  $p_1 = 0$ ,  $p_2 = 3i$  and  $p_3 = -4$ .

■

2. Compute

$$\int_{-1}^1 z^i dz$$

where the integrand denote the principal branch

$$z^i = \exp(i \operatorname{Log} z)$$

of  $z^i$  and where the path of integration is any continuous curve from  $z = -1$  to  $z = 1$  that, except for its starting and ending points, lies below the real axis.

*Solution.* Note that  $z^{i+1}/(i+1)$  is an anti-derivative of  $z^i$  outside the branch locus  $(-\infty, 0]$ . So

$$\begin{aligned}\int_{-1}^1 z^i dz &= \left. \frac{z^{i+1}}{i+1} \right|_1 - \lim_{\substack{z \rightarrow -1 \\ \text{Im}(z) < 0}} \frac{z^{i+1}}{i+1} \\ &= \frac{1}{i+1} - \frac{\exp((i+1)(-\pi i))}{i+1} \\ &= \frac{1+e^\pi}{i+1} = \frac{1+e^\pi}{2}(1-i)\end{aligned}$$

■

3. Apply Cauchy Integral Theorem to show that

$$\int_C f(z) dz = 0$$

when  $C$  is the unit circle  $|z| = 1$ , in either direction, and when

- (a)  $f(z) = \frac{z^3}{z^2 + 5z + 6}$ ;
- (b)  $f(z) = e^{\tan z}$ ;
- (c)  $f(z) = \text{Log}(z + 3i)$ .

*Solution.* By Cauchy Integral Theorem,  $\int_{|z|=1} f(z) dz = 0$  if  $f(z)$  is analytic on and inside the circle  $|z| = 1$ . Hence it is enough to show that  $f(z)$  is analytic in  $\{|z| \leq 1\}$ .

- (a)  $f(z)$  is analytic in  $\{z \neq -2, -3\}$  and hence analytic in  $\{|z| \leq 1\}$ .
- (b)  $f(z)$  is analytic in  $\{z : \cos z = 0\} = \{z = n\pi + \pi/2, n \in \mathbb{Z}\}$ . Since  $|n\pi + \pi/2| > 1$  for all integers  $n$ ,  $f(z)$  is analytic in  $\{|z| \leq 1\}$ .
- (c)  $\text{Log}(z)$  is analytic in  $\mathbb{C} \setminus (-\infty, 0]$  and hence  $\text{Log}(z + 3i)$  is analytic in  $\mathbb{C} \setminus \{z : z = x - 3i, x \in (-\infty, 0]\}$ . Since  $|x - 3i| > 1$  for all  $x$  real,  $f(z)$  is analytic in  $\{|z| \leq 1\}$ .

■

4. Let  $C_1$  denote the positively oriented boundary of the curve given by  $|x| + |y| = 2$  and  $C_2$  be the positively oriented circle  $|z| = 4$ . Apply Cauchy Integral Theorem to show that

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$$

when

- (a)  $f(z) = \frac{z+1}{z^2+1}$ ;
- (b)  $f(z) = \frac{z+2}{\sin(z/2)}$ ;
- (c)  $f(z) = \frac{\sin(z)}{z^2+6z+5}$ .

*Solution.* By Cauchy Integral Theorem,  $\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$  if  $f(z)$  is analytic on and between  $C_1$  and  $C_2$ . Hence it is enough to show that  $f(z)$  is analytic in  $\{|x| + |y| \geq 2, |z| \leq 4\}$ .

- (a)  $f(z)$  is analytic in  $\{z \neq \pm i\}$ . Since  $\pm i \in \{|x| + |y| < 2\}$ ,  $f(z)$  is analytic in  $\{|x| + |y| \geq 2, |z| \leq 4\}$ .
- (b)  $f(z)$  is analytic in  $\{z : \sin(z/2) \neq 0\} = \{z \neq 2n\pi : n \in \mathbb{Z}\}$ . Since  $2n\pi \in \{|x| + |y| < 2\}$  for  $n = 0$  and  $|2n\pi| > 4$  for  $n \neq 0$  and  $n \in \mathbb{Z}$ ,  $f(z)$  is analytic in  $\{|x| + |y| \geq 2, |z| \leq 4\}$ .
- (c)  $f(z)$  is analytic in  $\{z \neq -1, -5\}$ . Since  $-1 \in \{|x| + |y| < 2\}$  for  $n = 0$  and  $|-5| > 4$ ,  $f(z)$  is analytic in  $\{|x| + |y| \geq 2, |z| \leq 4\}$ .

■

5. Let  $C$  denote the positively oriented boundary of the square whose sides lie along the lines  $x = \pm 2$  and  $y = \pm 2$ . Evaluate each of these integrals

- (a)  $\int_C \frac{z dz}{z+1}$ ;
- (b)  $\int_C \frac{\cosh z}{z^2+z} dz$ ;
- (c)  $\int_C \frac{\tan(z/2)}{z-\pi/2} dz$ .

*Solution.* (a) By Cauchy Integral Formula,

$$\int_C \frac{zdz}{z+1} = 2\pi i(-1) = -2\pi i.$$

(b) By Cauchy Integral Theorem,

$$\int_C \frac{\cosh z}{z^2 + z} dz = \int_{|z|=r} \frac{\cosh z}{z^2 + z} dz + \int_{|z+1|=r} \frac{\cosh z}{z^2 + z} dz$$

for  $r = 1/2$ . By Cauchy Integral Formula,

$$\int_{|z|=r} \frac{\cosh z}{z^2 + z} dz = 2\pi i \left. \frac{\cosh(z)}{z+1} \right|_{z=0} = 2\pi i$$

and

$$\int_{|z+1|=r} \frac{\cosh z}{z^2 + z} dz = 2\pi i \left. \frac{\cosh z}{z} \right|_{z=-1} = -2\pi i \cosh(-1).$$

Hence

$$\int_C \frac{\cosh z}{z^2 + z} dz = 2\pi i(1 - \cosh(-1)).$$

(c) Note that  $\tan(z/2)$  is analytic in  $\{z \neq (2n+1)\pi : n \in \mathbb{Z}\}$  and hence analytic inside  $C$ . Therefore,

$$\int_C \frac{\tan(z/2)}{z - \pi/2} dz = 2\pi i \tan(\pi/4) = 2\pi i$$

by Cauchy Integral Formula. ■

6. Find the value of the integral  $g(z)$  around the circle  $|z - i| = 2$  oriented counterclockwise when

(a)  $g(z) = \frac{1}{z^2 + 4};$

(b)  $g(z) = \frac{1}{z(z^2 + 4)}.$

*Solution.* (a) Since  $-2i \notin \{|z - i| \leq 2\}$  and  $2i \in \{|z - i| \leq 2\}$ ,

$$\int_{|z-i|=2} g(z) dz = \int_{|z-i|=2} \frac{(z+2i)^{-1}}{z-2i} dz = 2\pi i(2i+2i)^{-1} = \frac{\pi}{2}$$

by Cauchy Integral Formula.



(b) By Cauchy Integral Theorem,

$$\int_{|z-i|=2} g(z)dz = \int_{|z|=r} g(z)dz + \int_{|z-2i|=r} g(z)dz$$

for  $r < 1/2$ . Since

$$\int_{|z|=r} g(z)dz = 2\pi i \left. \frac{1}{z^2 + 4} \right|_{z=0} = \frac{\pi i}{2}$$

and

$$\int_{|z-2i|=r} g(z)dz = 2\pi i \left. \frac{1}{z(z+2i)} \right|_{z=2i} = -\frac{\pi i}{4}$$

by Cauchy Integral Formula,

$$\int_{|z-i|=2} g(z)dz = \frac{\pi i}{4}$$

■

7. Compute the integrals of the following functions along the curves  $C_1 = \{|z| = 1\}$  and  $C_2 = \{|z - 2| = 1\}$ , both oriented counterclockwise:

(a)  $\frac{1}{2z - z^2};$

(b)  $\frac{\sinh z}{(2z - z^2)^2}.$

*Solution.* (a)

$$\int_{|z|=1} \frac{dz}{2z - z^2} = \int_{|z|=1} \frac{(2 - z)^{-1}}{z} dz = 2\pi i (2 - 0)^{-1} = \pi i$$

(b)

$$\begin{aligned} \int_{|z|=1} \frac{\sinh z}{(2z - z^2)^2} dz &= \int_{|z|=1} \frac{(\sinh z)(2 - z)^{-2}}{z^2} dz \\ &= 2\pi i \left( (\sinh z)(2 - z)^{-2} \right)' \Big|_{z=0} = \frac{\pi i}{2} \end{aligned}$$

■

8. Show that if  $f$  is analytic inside and on a simple closed curve  $C$  and  $z_0$  is not on  $C$ , then

$$(n-1)! \int_C \frac{f^{(m)}(z)}{(z-z_0)^n} dz = (m+n-1)! \int_C \frac{f(z)}{(z-z_0)^{m+n}} dz$$

for all positive integers  $m$  and  $n$ .

*Proof.* If  $z_0$  lies outside  $C$ , then

$$\int_C \frac{f^{(m)}(z)}{(z-z_0)^n} dz = \int_C \frac{f(z)}{(z-z_0)^{m+n}} dz = 0$$

by Cauchy Integral Theorem, since  $f^{(m)}z/(z-z_0)^n$  and  $f(z)/(z-z_0)^{m+n}$  are analytic on and inside  $C$ .

If  $z_0$  lies inside  $C$ , then

$$(n-1)! \int_C \frac{f^{(m)}(z)}{(z-z_0)^n} dz = (f^{(m)}(z))^{(n-1)} \Big|_{z=z_0} = f^{(m+n-1)}(z_0)$$

and

$$(m+n-1)! \int_C \frac{f(z)}{(z-z_0)^{m+n}} dz = f^{(m+n-1)}(z_0)$$

by Cauchy Integral Formula. Therefore,

$$(n-1)! \int_C \frac{f^{(m)}(z)}{(z-z_0)^n} dz = (m+n-1)! \int_C \frac{f(z)}{(z-z_0)^{m+n}} dz.$$

■

9. Let  $f(z)$  be an entire function. Show that  $f(z)$  is a constant if  $|f(z)| \leq \ln(|z| + 1)$  for all  $z \in \mathbb{C}$ .

*Proof.* For every  $z_0 \in \mathbb{C}$ , we have

$$f'(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2} dz$$

for all  $R > 0$ . Since

$$\left| \frac{f(z)}{(z-z_0)^2} \right| \leq \frac{\ln(|z| + 1)}{R^2} \leq \frac{\ln(R + |z_0| + 1)}{R^2}$$

for  $|z - z_0| = R$ ,

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z-z_0)^2} dz \right| \leq \frac{\ln(R + |z_0|) + 1}{R}.$$

And since

$$\lim_{R \rightarrow \infty} \frac{\ln(R + |z_0| + 1)}{R} = \lim_{R \rightarrow \infty} \frac{1}{R + |z_0| + 1} = 0,$$

by L'Hospital (see Problem 5, section 2.2), we conclude that  $|f'(z_0)| = 0$  and hence  $f'(z_0) = 0$  for every  $z_0 \in \mathbb{C}$ . Therefore,  $f(z)$  is a constant. ■

10. Let  $C_N$  be the boundary of the square

$$\{|x| \leq N\pi, |y| \leq N\pi\},$$

where  $N$  is a positive integer. Show that

$$\lim_{N \rightarrow \infty} \int_{C_N} \frac{dz}{z^3 \cos z} = 0$$

*Proof.* When  $z = x + yi \in C_N$ , either  $x = \pm N\pi$  or  $y = \pm N\pi$ . When  $x = \pm N\pi$ ,

$$|\cos z|^2 = (\cos x)^2 + (\sinh y)^2 \geq (\cos x)^2 = (\cos(N\pi))^2 = 1$$

When  $y = \pm N\pi$ ,

$$|\cos z|^2 = (\cos x)^2 + (\sinh y)^2 \geq (\sinh y)^2 = (\sinh(N\pi))^2 > 1$$

Therefore,  $|\cos z| \geq 1$  when  $z \in C_N$ . We also have  $|z| \geq N\pi$  when  $z \in C_N$ . Therefore,

$$\left| \frac{1}{z^3 \cos z} \right| \leq \frac{1}{N^3 \pi^3}$$

and

$$\left| \int_{C_N} \frac{dz}{z^3 \cos z} \right| \leq \frac{1}{N^3 \pi^3} \int_{C_N} |dz| = \frac{8N\pi}{N^3 \pi^3} = \frac{8}{N^2 \pi^2}$$

Since

$$\lim_{N \rightarrow \infty} \frac{8}{N^2 \pi^2} = 0$$

we conclude

$$\lim_{N \rightarrow \infty} \int_{C_N} \frac{dz}{z^3 \cos z} = 0$$

■

11. Let  $C_N$  be the boundary of the square

$$\left\{ |x| \leq N\pi + \frac{\pi}{2}, |y| \leq N\pi + \frac{\pi}{2} \right\}$$

oriented counterclockwise, where  $N$  is a positive integer. Show that

$$\lim_{N \rightarrow \infty} \int_{C_N} \frac{dz}{z^2 \sin z} = 0.$$

[Refer to: problem 5 in section 4.3]

*Proof.* When  $z = x + yi \in C_N$ , either  $x = \pm(N\pi + \pi/2)$  or  $y = \pm(N\pi + \pi/2)$ . When  $x = \pm(N\pi + \pi/2)$ ,

$$|\sin z|^2 = (\sin x)^2 + (\sinh y)^2 \geq (\sin x)^2 = (\sin(N\pi + \pi/2))^2 = 1$$

When  $y = \pm(N\pi + \pi/2)$ ,

$$\begin{aligned} |\sin z|^2 &= (\sin x)^2 + (\sinh y)^2 \geq (\sinh y)^2 \\ &= (\sinh(N\pi + \pi/2))^2 \geq (\sinh(3\pi/2))^2 > 1. \end{aligned}$$

Therefore,  $|\sin z| \geq 1$  when  $z \in C_N$ . We also have  $|z| \geq N\pi + \pi/2$  when  $z \in C_N$ . Consequently,

$$\left| \frac{1}{z^2 \sin z} \right| \leq \frac{1}{(N + 1/2)^2 \pi^2}$$

and

$$\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \leq \frac{1}{(N + 1/2)^2 \pi^2} \int_{C_N} |dz| = \frac{8(N + 1/2)\pi}{(N + 1/2)^2 \pi^2} = \frac{8}{(N + 1/2)\pi}.$$

And since

$$\lim_{N \rightarrow \infty} \frac{8}{(N + 1/2)\pi} = 0$$

we conclude

$$\lim_{N \rightarrow \infty} \int_{C_N} \frac{dz}{z^2 \sin z} = 0$$

■

12. Compute the contour integral

$$\int_C \frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1} dz,$$

where  $C$  is the circle  $|z| = 2$  oriented counter-clockwise.

*Solution.* First, we show that all roots of

$$z^{2011} + z^{2010} + z^{2009} + 1 = 0$$

lie inside  $|z| < 2$ . Otherwise, suppose that

$$z^{2011} + z^{2010} + z^{2009} + 1 = 0$$

for some  $|z| \geq 2$ . Then

$$1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^{2011}} = 0$$

and hence

$$1 = \left| -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^{2011}} \right| \leq \frac{1}{|z|} + \frac{1}{|z|^2} + \frac{1}{|z|^{2011}}.$$

When  $|z| \geq 2$ ,

$$\frac{1}{|z|} + \frac{1}{|z|^2} + \frac{1}{|z|^{2011}} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{2^{2011}} < 1.$$

This is a contradiction. Therefore, all roots of

$$z^{2011} + z^{2010} + z^{2009} + 1 = 0$$

lie inside  $|z| < 2$ . It follows that

$$\int_C \frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1} dz = \int_{|z|=R} \frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1} dz$$

for all  $R > 2$  by CIT.

We have

$$\frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1} = 1 - \frac{1}{z} + \frac{z^{2009} - z + 1}{z(z^{2011} + z^{2010} + z^{2009} + 1)}.$$

Since

$$\left| \frac{z^{2009} - z + 1}{z(z^{2011} + z^{2010} + z^{2009} + 1)} \right| \leq \frac{R^{2009} + R + 1}{R(R^{2011} - R^{2010} - R^{2009} - 1)}$$

for  $|z| = R$ ,

$$\left| \int_{|z|=R} \frac{z^{2009} - z + 1}{z(z^{2011} + z^{2010} + z^{2009} + 1)} dz \right| \leq \frac{2\pi(R^{2009} + R + 1)}{R^{2011} - R^{2010} - R^{2009} - 1}$$

and hence

$$\lim_{R \rightarrow \infty} \int_{|z|=R} \frac{z^{2009} - z + 1}{z(z^{2011} + z^{2010} + z^{2009} + 1)} dz = 0.$$

And we have

$$\int_{|z|=R} dz = 0 \text{ and } \int_{|z|=R} \frac{dz}{z} = 2\pi i.$$

Therefore,

$$\int_C \frac{z^{2011}}{z^{2011} + z^{2010} + z^{2009} + 1} dz = -2\pi i.$$

■

13. Calculate

$$\int_C \frac{z^{2008}}{z^{2009} + z + 1} dz$$

where  $C$  is the circle  $|z| = 2$  oriented counter-clockwise.

*Solution.* First, we prove that all zeroes of  $z^{2009} + z + 1$  lie inside the circle  $|z| = 2$ . Otherwise,  $z^{2009} + z + 1 = 0$  for some  $|z| \geq 2$ . Then

$$z^{2009} + z + 1 = 0 \Rightarrow 1 + \frac{1}{z^{2008}} + \frac{1}{z^{2009}} = 0$$

On the other hand,

$$\left| 1 + \frac{1}{z^{2008}} + \frac{1}{z^{2009}} \right| \geq 1 - \frac{1}{|z|^{2008}} - \frac{1}{|z|^{2009}} \geq 1 - \frac{1}{2^{2008}} - \frac{1}{2^{2009}} > 0$$

for  $|z| \geq 2$ . Contradiction. So all zeroes of  $z^{2009} + z + 1$  lie inside the circle  $|z| = 2$  and hence  $z^{2008} / (z^{2009} + z + 1)$  is analytic in  $|z| \geq 2$ . Therefore,

$$\int_{|z|=2} \frac{z^{2008}}{z^{2009} + z + 1} dz = \int_{|z|=R} \frac{z^{2008}}{z^{2009} + z + 1} dz$$

for all  $R > 2$  by Cauchy Integral Theorem.

We observe that

$$\frac{z^{2008}}{z^{2009} + z + 1} - \frac{1}{z} = -\frac{z + 1}{z(z^{2009} + z + 1)}.$$

For  $|z| = R > 2$ ,

$$\left| \frac{z + 1}{z(z^{2009} + z + 1)} \right| \leq \frac{R + 1}{R(R^{2009} - R - 1)}$$

and hence

$$\left| \int_{|z|=R} \frac{z+1}{z^{2009}+z+1} dz \right| \leq \frac{2\pi(R+1)}{R^{2009}-R-1}.$$

It follows that

$$\lim_{R \rightarrow \infty} \int_{|z|=R} \frac{z+1}{z^{2009}+z+1} dz = 0.$$

Therefore,

$$\begin{aligned} \int_{|z|=2} \frac{z^{2008}}{z^{2009}+z+1} dz &= \lim_{R \rightarrow \infty} \int_{|z|=R} \frac{z^{2008}}{z^{2009}+z+1} dz \\ &= \lim_{R \rightarrow \infty} \int_{|z|=R} \frac{dz}{z} = 2\pi i. \end{aligned}$$

■

14. Let  $C$  be the circle  $|z| = 1$  oriented counter-clockwise.

(a) Compute

$$\int_C \frac{1}{z^2 - 8z + 1} dz$$

(b) Use or not use part (a) to compute

$$\int_0^\pi \frac{1}{4 - \cos \theta} d\theta$$

*Solution.* The function

$$\frac{1}{z^2 - 8z + 1} = \frac{1}{(z - 4 - \sqrt{15})(z - 4 + \sqrt{15})}$$

has a singularity in  $|z| < 1$  at  $z = 4 - \sqrt{15}$ . Therefore,

$$\begin{aligned} \int_C \frac{1}{z^2 - 8z + 1} dz &= 2\pi i \lim_{z \rightarrow 4 - \sqrt{15}} \frac{1}{(z - 4 - \sqrt{15})(z - 4 + \sqrt{15})} \\ &= 2\pi i \left( \frac{1}{z - 4 - \sqrt{15}} \right) \Big|_{z=4-\sqrt{15}} = -\frac{\pi i}{\sqrt{15}} \end{aligned}$$

That is,

$$\begin{aligned}
 \int_C \frac{1}{z^2 - 8z + 1} dz &= \int_{-\pi}^{\pi} \frac{de^{i\theta}}{e^{2i\theta} - 8e^{i\theta} + 1} \\
 &= i \int_{-\pi}^{\pi} \frac{e^{i\theta}}{e^{2i\theta} - 8e^{i\theta} + 1} d\theta \\
 &= i \int_{-\pi}^{\pi} \frac{1}{e^{i\theta} + e^{-i\theta} - 8} d\theta \\
 &= -\frac{i}{2} \int_{-\pi}^{\pi} \frac{1}{4 - \cos \theta} d\theta \\
 &= -i \int_0^{\pi} \frac{1}{4 - \cos \theta} d\theta
 \end{aligned}$$

Therefore,

$$\int_0^{\pi} \frac{1}{4 - \cos \theta} d\theta = \frac{\pi}{\sqrt{15}}$$

■

15. Compute the integral

$$\int_{-\pi}^{\pi} \frac{dx}{2 - (\cos x + \sin x)}.$$

*Solution.* Let  $z = e^{ix}$ . Then  $dz = ie^{ix} dx$ ,  $dx = -idz/z$  and hence

$$\begin{aligned}
 \int_{-\pi}^{\pi} \frac{dx}{2 - (\cos x + \sin x)} &= \int_{-\pi}^{\pi} \frac{dx}{2 - (e^{ix} + e^{-ix})/2 - (e^{ix} - e^{-ix})/(2i)} \\
 &= \int_{|z|=1} \frac{-idz}{2z - (z^2 + 1)/2 - (z^2 - 1)/(2i)} \\
 &= (i - 1) \int_{|z|=1} \frac{dz}{z^2 - 2(1 + i)z + i} \\
 &= (i - 1) \int_{|z|=1} \frac{dz}{(z - z_1)(z - z_2)} \\
 &= \frac{2\pi i(i - 1)}{z_2 - z_1} = \sqrt{2}\pi
 \end{aligned}$$

where  $z_1 = (1 + \sqrt{2}/2) + (1 + \sqrt{2}/2)i$  and  $z_2 = (1 - \sqrt{2}/2) + (1 - \sqrt{2}/2)i$ .

■



16. Let  $f(z)$  be an entire function satisfying

$$|f(z_1 + z_2)| \leq |f(z_1)| + |f(z_2)|$$

for all complex numbers  $z_1$  and  $z_2$ . Show that  $f(z)$  is a polynomial of degree at most 1.

*Proof.* We have

$$\begin{aligned} \sum_{k=1}^n f(z_k) &= f(z_1) + f(z_2) + \sum_{k=3}^n f(z_k) \\ &= f(z_1 + z_2) + \sum_{k=3}^n f(z_k) \\ &= f(z_1 + z_2) + f(z_3) + \sum_{k=4}^n f(z_k) \\ &= f(z_1 + z_2 + z_3) + \sum_{k=4}^n f(z_k) \\ &= \dots = f(z_1 + z_2 + \dots + z_n) = f\left(\sum_{k=1}^n x_k\right). \end{aligned}$$

Therefore,

$$\sum_{k=1}^n f(z_k) = f(z_1) + f(z_2) + \dots + f(z_n) = f(z_1 + z_2 + \dots + z_n) = f\left(\sum_{k=1}^n x_k\right)$$

for all complex numbers  $z_1, z_2, \dots, z_n$ . Particularly, this holds for  $z_1 = z_2 = \dots = z_n = z/n$ :

$$nf\left(\frac{z}{n}\right) = f(z)$$

for all  $z \in \mathbb{C}$  and all positive integer  $n$ . Let  $M$  be the maximum of  $|f(z)|$  for  $|z| = 1$ . Then

$$|f(z)| = n \left| f\left(\frac{z}{n}\right) \right| \leq nM$$

for all  $z$  satisfying  $|z| = n$ .

By Cauchy Integral Formula,

$$f''(z_0) = \frac{1}{\pi i} \int_{|z|=n} \frac{f(z)}{(z - z_0)^3} dz$$

for  $|z_0| < n$ . Since

$$\left| \frac{f(z)}{(z - z_0)^3} \right| = \frac{|f(z)|}{|z - z_0|^3} \leq \frac{nM}{(n - |z_0|)^3}$$

for  $|z| = n$  and  $|z_0| < n$ ,

$$\left| \frac{1}{\pi i} \int_{|z|=n} \frac{f(z)}{(z - z_0)^3} dz \right| \leq \frac{2n^2 M}{(n - |z_0|)^3}.$$

And since

$$\lim_{n \rightarrow \infty} \frac{2n^2 M}{(n - |z_0|)^3} = \lim_{n \rightarrow \infty} \frac{2M/n}{(1 - |z_0|/n)^3} = 0,$$

we conclude that  $f''(z_0) = 0$  for all  $z_0$ . Therefore,  $f'(z) \equiv a$  is a constant and  $f(z) = az + b$  is a polynomial of degree at most 1. ■

17. Let  $f(z)$  be an entire function satisfying that  $|f(z)| \leq |z|^2$  for all  $z$ . Show that  $f(z) \equiv az^2$  for some constant  $a$  satisfying  $|a| \leq 1$ .

*Proof.* For every  $z_0 \in \mathbb{C}$ , we have

$$f'''(z_0) = \frac{3!}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z - z_0)^4} dz$$

for all  $R > 0$ . Since

$$\left| \frac{f(z)}{(z - z_0)^4} \right| \leq \frac{|z|^2}{R^4} \leq \frac{(R + |z_0|)^2}{R^4}$$

for  $|z - z_0| = R$ ,

$$|f'''(z_0)| = \left| \frac{3!}{2\pi i} \int_{|z-z_0|=R} \frac{f(z)}{(z - z_0)^4} dz \right| \leq \frac{6(R + |z_0|)^2}{R^3}.$$

And since

$$\lim_{R \rightarrow \infty} \frac{6(R + |z_0|)^2}{R^3} = \lim_{R \rightarrow \infty} \frac{6}{R} \left( 1 + \frac{|z_0|}{R} \right)^2 = 0,$$

we conclude that  $|f'''(z_0)| = 0$  and hence  $f'''(z_0) = 0$  for every  $z_0 \in \mathbb{C}$ . Therefore,  $f'''(z) \equiv 0$ ,  $f''(z) \equiv 2a$ ,  $f'(z) \equiv 2az + b$  and  $f(z) \equiv az^2 + bz + c$  for some constants  $a, b$  and  $c$ .

Since  $|f(z)| \leq |z|^2$ ,  $|az^2 + bz + c| \leq |z|^2$  for all  $z$ . Take  $z = 0$  and we obtain  $|c| \leq 0$ . Hence  $c = 0$ . Therefore,  $|az^2 + bz| \leq |z|^2$  and hence  $|az + b| \leq |z|$  for all  $z$ . Take  $z = 0$  again and we obtain  $|b| \leq 0$ . Hence  $b = 0$ . So  $|az^2| \leq |z|^2$  and hence  $|a| \leq 1$ . In conclusion,  $f(z) = az^2$  with  $a$  satisfying  $|a| \leq 1$ . ■

18. Let  $f(z)$  be a complex polynomial of degree at least 2 and  $R$  be a positive number such that  $f(z) \neq 0$  for all  $|z| \geq R$ . Show that

$$\int_{|z|=R} \frac{dz}{f(z)} = 0.$$

[Refer to: problem 4 in section 4.3]

*Proof.* Let  $f(z) = a_0 + a_1z + \cdots + a_nz^n$ , where  $a_n \neq 0$  and  $n = \deg f$ . Since  $f(z) \neq 0$  for  $|z| \geq R$ ,  $1/f(z)$  is analytic in  $|z| \geq R$ . Hence

$$\int_{|z|=R} \frac{dz}{f(z)} = \int_{|z|=r} \frac{dz}{f(z)}$$

for all  $r \geq R$  by Cauchy Integral Theorem.

Since

$$|f(z)| \geq |a_n||z|^n - |a_{n-1}||z|^{n-1} - \cdots - |a_0|$$

we have

$$\left| \frac{1}{f(z)} \right| \leq \frac{1}{|a_n|r^n - |a_{n-1}|r^{n-1} - \cdots - |a_0|}$$

for  $|z| = r$  sufficiently large. It follows that

$$\left| \int_{|z|=r} \frac{dz}{f(z)} \right| \leq \frac{2\pi r}{|a_n|r^n - |a_{n-1}|r^{n-1} - \cdots - |a_0|}$$

And since  $n \geq 2$ ,

$$\begin{aligned} & \lim_{r \rightarrow \infty} \frac{2\pi r}{|a_n|r^n - |a_{n-1}|r^{n-1} - \cdots - |a_0|} \\ &= \lim_{r \rightarrow \infty} \frac{2\pi}{n|a_n|r^{n-1} - (n-1)|a_{n-1}|r^{n-1} - \cdots - |a_1|} = 0 \end{aligned}$$

by L'Hospital. Hence

$$\int_{|z|=R} \frac{dz}{f(z)} = \lim_{r \rightarrow \infty} \int_{|z|=r} \frac{dz}{f(z)} = 0.$$

■

### 3.3 Improper integrals

1. Compute the integral

$$\int_0^\infty \frac{x dx}{x^3 + 1}.$$

*Solution.* Consider the contour integral of  $z/(z^3 + 1)$  along  $L_R = [0, R]$ ,  $C_R = \{z = Re^{it} : 0 \leq t \leq 2\pi/3\}$  and  $M_R = \{te^{2\pi i/3} : 0 \leq t \leq R\}$ . By Cauchy Integral Formula,

$$\int_{L_R} \frac{z dz}{z^3 + 1} + \int_{C_R} \frac{z dz}{z^3 + 1} - \int_{M_R} \frac{z dz}{z^3 + 1} = \int_{|z - e^{\pi i/3}|=1/2} \frac{z dz}{z^3 + 1}$$

By Cauchy Integral Formula,

$$\begin{aligned} \int_{|z - e^{\pi i/3}|=1/2} \frac{z dz}{z^3 + 1} &= \frac{2\pi i \exp(\pi i/3)}{(\exp(\pi i/3) + 1)(\exp(\pi i/3) - \exp(-\pi i/3))} \\ &= \frac{2\pi \exp(\pi i/3)}{(\exp(\pi i/3) + 1)\sqrt{3}}. \end{aligned}$$

For  $z$  lying on  $C_R$ ,

$$\left| \frac{z}{z^3 + 1} \right| \leq \frac{R}{R^3 - 1}$$

and hence

$$\left| \int_{C_R} \frac{z dz}{z^3 + 1} \right| \leq \frac{2\pi R}{3(R^3 - 1)}$$

It follows that

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{z dz}{z^3 + 1} = 0$$

And

$$\int_{M_R} \frac{z dz}{z^3 + 1} = \exp(4\pi i/3) \int_0^R \frac{x dx}{x^3 + 1}$$

Therefore, we have

$$(1 - \exp(4\pi i/3)) \int_0^\infty \frac{x dx}{x^3 + 1} = \frac{2\pi \exp(\pi i/3)}{(\exp(\pi i/3) + 1)\sqrt{3}}$$

and hence

$$\int_0^\infty \frac{x dx}{x^3 + 1} = \frac{2\pi}{3\sqrt{3}}.$$



2. Compute the integral

$$\int_0^\infty \frac{\cos x}{x^4 + 1} dx.$$

*Solution.* Since  $\cos x / (x^4 + 1)$  is even,

$$\int_0^\infty \frac{\cos x}{x^4 + 1} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos x}{x^4 + 1} dx.$$

Actually, we have

$$\int_0^\infty \frac{\cos x}{x^4 + 1} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{e^{ix}}{x^4 + 1} dx$$

since  $e^{ix} = \cos x + i \sin x$ .

Consider the contour integral of  $e^{iz} / (z^4 + 1)$  along the path  $L_R = [-R, R]$  and  $C_R = \{|z| = R, \operatorname{Im}(z) \geq 0\}$ , oriented counterclockwise. By Cauchy Integral Theorem, we have

$$\begin{aligned} & \int_{L_R} \frac{e^{iz}}{z^4 + 1} dz + \int_{C_R} \frac{e^{iz}}{z^4 + 1} dz \\ &= \int_{|z - e^{\pi i/4}|=1/2} \frac{e^{iz}}{z^4 + 1} dz + \int_{|z - e^{3\pi i/4}|=1/2} \frac{e^{iz}}{z^4 + 1} dz \end{aligned}$$

By Cauchy Integral Formula,

$$\begin{aligned} \int_{|z - e^{\pi i/4}|=1/2} \frac{e^{iz}}{z^4 + 1} dz &= \frac{2\pi i e^{i(\sqrt{2}+i\sqrt{2})/2}}{(e^{\pi i/4} - e^{3\pi i/4})(e^{\pi i/4} - e^{-\pi i/4})(e^{\pi i/4} - e^{-3\pi i/4})} \\ &= \frac{\pi(1-i) \exp((- \sqrt{2} + i\sqrt{2})/2)}{2\sqrt{2}} \end{aligned}$$

and similarly,

$$\int_{|z - e^{3\pi i/4}|=1/2} \frac{e^{iz}}{z^4 + 1} dz = \frac{\pi(1+i) \exp((- \sqrt{2} - i\sqrt{2})/2)}{2\sqrt{2}}$$

Therefore,

$$\int_{L_R} \frac{e^{iz}}{z^4 + 1} dz + \int_{C_R} \frac{e^{iz}}{z^4 + 1} dz = \frac{\pi e^{-\sqrt{2}/2}}{\sqrt{2}} (\cos(\sqrt{2}/2) + \sin(\sqrt{2}/2)).$$

For  $z$  lying on  $C_R$ ,  $y = \operatorname{Im}(z) \geq 0$  and hence  $|e^{iz}| = e^{-y} \leq 1$ . Hence

$$\left| \frac{e^{iz}}{z^4 + 1} \right| \leq \frac{1}{R^4 - 1}$$

and it follows that

$$\left| \int_{C_R} \frac{e^{iz}}{z^4 + 1} dz \right| \leq \frac{\pi R}{R^4 - 1}$$

Since

$$\lim_{R \rightarrow \infty} \frac{\pi R}{R^4 - 1} = 0,$$

we conclude that

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{iz}}{z^4 + 1} dz = 0.$$

Therefore,

$$\begin{aligned} \int_0^\infty \frac{\cos x}{x^4 + 1} dx &= \frac{1}{2} \int_{-\infty}^\infty \frac{e^{ix}}{x^4 + 1} dx \\ &= \frac{1}{2} \lim_{R \rightarrow \infty} \int_{L_R} \frac{e^{iz}}{z^4 + 1} dz \\ &= \frac{\pi e^{-\sqrt{2}/2}}{2\sqrt{2}} \left( \cos \left( \frac{\sqrt{2}}{2} \right) + \sin \left( \frac{\sqrt{2}}{2} \right) \right). \end{aligned}$$

■



# Chapter 4

## Series

### 4.1 Taylor and Laurent series

1. Find the Taylor series of the following functions and their radii of convergence:

(a)  $z \sinh(z^2)$  at  $z = 0$ ;

(b)  $e^z$  at  $z = 2$ ;

(c)  $\frac{z^2 + z}{(1 - z)^2}$  at  $z = -1$ .

*Solution.* (a) Since  $e^z = \sum_{n=0}^{\infty} z^n / n!$ ,

$$\begin{aligned} z \sinh(z^2) &= z \left( \frac{e^{z^2} - e^{-z^2}}{2} \right) \\ &= \frac{z}{2} \left( \sum_{n=0}^{\infty} \frac{z^{2n}}{n!} - \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{n!} \right) \\ &= \sum_{n=0}^{\infty} \frac{1 - (-1)^n}{2} \frac{z^{2n+1}}{n!} \\ &= \sum_{m=0}^{\infty} \frac{z^{4m+3}}{(2m+1)!} \end{aligned}$$

where we observe that  $(1 - (-1)^n)/2 = 0$  if  $n = 2m$  is even and 1 if  $n = 2m + 1$  is odd. Since  $f(z)$  is entire, the radius of convergence is  $\infty$ .



(b) Let  $w = z - 2$ . Then  $z = w + 2$  and

$$e^z = e^{w+2} = e^2 e^w = e^2 \sum_{n=0}^{\infty} \frac{w^n}{n!} = \sum_{n=0}^{\infty} \frac{e^2 (z-2)^n}{n!}.$$

Since  $f(z)$  is entire, the radius of convergence is  $\infty$ .

(c) Let  $w = z + 1$ . Then

$$\frac{z^2 + z}{(1-z)^2} = \frac{w^2 - w}{(2-w)^2} = 1 - \frac{3}{2-w} + \frac{2}{(2-w)^2}.$$

We have

$$-\frac{3}{2-w} = -\frac{3}{2} \frac{1}{1-(w/2)} = -\frac{3}{2} \sum_{n=0}^{\infty} \frac{w^n}{2^n} = -\sum_{n=0}^{\infty} \frac{3w^n}{2^{n+1}}$$

and

$$\begin{aligned} \frac{2}{(2-w)^2} &= \left( \frac{2}{2-w} \right)' = \left( \frac{1}{1-(w/2)} \right)' \\ &= \left( \sum_{n=0}^{\infty} \frac{w^n}{2^n} \right)' = \sum_{n=0}^{\infty} \frac{n w^{n-1}}{2^n} \\ &= \sum_{n=0}^{\infty} \frac{(n+1)w^n}{2^{n+1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{z^2 + z}{(1-z)^2} &= 1 - \sum_{n=0}^{\infty} \frac{3w^n}{2^{n+1}} + \sum_{n=0}^{\infty} \frac{(n+1)w^n}{2^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{(n-2)(z+1)^n}{2^{n+1}}. \end{aligned}$$

Since  $f(z)$  is analytic in  $|z+1| < 2$  and has a singularity at  $z = 1$ , the radius of convergence is 2. ■

2. Find the Taylor series of  $(\cos z)^2$  at  $z = \pi$ .

*Solution.* Let  $w = z - \pi$ . Then

$$\begin{aligned}
 (\cos z)^2 &= (\cos(z + \pi))^2 = (\cos w)^2 \\
 &= \left( \frac{e^{iw} + e^{-iw}}{2} \right)^2 = \frac{1}{4}(e^{2iw} + e^{-2iw} + 2) \\
 &= \frac{1}{4} \sum_{n=0}^{\infty} \frac{(2i)^n w^n}{n!} + \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-2i)^n w^n}{n!} + \frac{1}{2} \\
 &= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2i)^{2n} w^{2n}}{(2n)!} = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1} w^{2n}}{(2n)!} \\
 &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} w^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} (z - \pi)^{2n}}{(2n)!}
 \end{aligned}$$

■

3. Let  $f(z)$  be a function analytic at 0 and  $g(z) = f(z^2)$ . Show that  $g^{(2n-1)}(0) = 0$  for all positive integers  $n$ .

*Proof.* Since  $f(z)$  is analytic at 0,  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  in some disk  $|z| < r$ . Therefore,  $g(z) = f(z^2) = \sum_{n=0}^{\infty} a_n z^{2n}$  in  $|z| < \sqrt{r}$  and hence

$$\sum_{m=0}^{\infty} \frac{g^{(m)}(0)}{m!} z^m = \sum_{n=0}^{\infty} a_n z^{2n}$$

And since the power series representation of an analytic function is unique, we must have  $g^{(m)}(0) = 0$  for  $m$  is odd, i.e.,  $m = 2n - 1$  for all positive integers  $n$ .

■

4. Find the Laurent series of the function

$$f(z) = \frac{z + 4}{z^2(z^2 + 3z + 2)}$$

in

- (a)  $0 < |z| < 1$ ;
- (b)  $1 < |z| < 2$ ;
- (c)  $|z| > 2$ ;
- (d)  $0 < |z + 1| < 1$ .

*Solution.* We write  $f(z)$  as a sum of partial fractions:

$$\frac{z+4}{z^2(z^2+3z+2)} = -\frac{5}{2z} + \frac{2}{z^2} + \frac{3}{z+1} - \frac{1}{2(z+2)}.$$

For  $0 < |z| < 1$ ,

$$\frac{3}{z+1} = 3 \sum_{n=0}^{\infty} (-1)^n z^n$$

and

$$-\frac{1}{2(z+2)} = -\frac{1}{4} \frac{1}{1+(z/2)} = -\frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{2^n}.$$

Therefore,

$$\begin{aligned} f(z) &= -\frac{5}{2z} + \frac{2}{z^2} + 3 \sum_{n=0}^{\infty} (-1)^n z^n - \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{2^n} \\ &= \frac{2}{z^2} - \frac{5}{2z} + \sum_{n=0}^{\infty} (-1)^n (3 - 2^{-n-2}) z^n \end{aligned}$$

For  $1 < |z| < 2$ ,

$$\frac{3}{z+1} = \frac{3}{z} \frac{1}{1+(1/z)} = \frac{3}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n}$$

and

$$-\frac{1}{2(z+2)} = -\frac{1}{4} \frac{1}{1+(z/2)} = -\frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{2^n}.$$

Therefore,

$$\begin{aligned} f(z) &= -\frac{5}{2z} + \frac{2}{z^2} + \frac{3}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} - \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{2^n} \\ &= \sum_{n=2}^{\infty} \frac{3(-1)^n}{z^{n+1}} - \frac{1}{z^2} + \frac{1}{2z} - \sum_{n=0}^{\infty} (-1)^n 2^{-n-2} z^n. \end{aligned}$$

For  $2 < |z| < \infty$ ,

$$\frac{3}{z+1} = \frac{3}{z} \frac{1}{1+(1/z)} = \frac{3}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n}$$

and

$$-\frac{1}{2(z+2)} = -\frac{1}{2z} \frac{1}{1+(2/z)} = -\frac{1}{2z} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{z^n}.$$

Therefore,

$$\begin{aligned} f(z) &= -\frac{5}{2z} + \frac{2}{z^2} + \frac{3}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} - \frac{1}{2z} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{z^n} \\ &= \sum_{n=3}^{\infty} \frac{(-1)^{n+1} (3 - 2^{n-2})}{z^n}. \end{aligned}$$

For  $0 < |z + 1| < 1$ , we let  $w = z + 1$  and then

$$\frac{z + 4}{z^2(z^2 + 3z + 2)} = \frac{5}{2(1 - w)} + \frac{2}{(1 - w)^2} + \frac{3}{w} - \frac{1}{2(w + 1)}.$$

For  $0 < |w| < 1$ ,

$$\begin{aligned} \frac{5}{2(1 - w)} &= \frac{5}{2} \sum_{n=0}^{\infty} w^n, \\ \frac{2}{(1 - w)^2} &= \left( \frac{2}{1 - w} \right)' = \left( \sum_{n=0}^{\infty} 2w^n \right)' = \sum_{n=0}^{\infty} 2(n + 1)w^n \end{aligned}$$

and

$$-\frac{1}{2(w + 1)} = -\frac{1}{2} \sum_{n=0}^{\infty} (-1)^n w^n.$$

Therefore,

$$\begin{aligned} f(z) &= \frac{3}{w} + \sum_{n=0}^{\infty} \left( 2n + \frac{9}{2} - \frac{(-1)^n}{2} \right) w^n \\ &= \frac{3}{z + 1} + \sum_{n=0}^{\infty} \left( 2n + \frac{9}{2} - \frac{(-1)^n}{2} \right) (z + 1)^n. \end{aligned}$$

■

5. Write the two Laurent series in powers of  $z$  that represent the function

$$f(z) = \frac{1}{z(1 + z^2)}$$

in certain domains and specify these domains.

*Solution.* Since  $f(z)$  is analytic at  $z \neq 0, \pm i$ , it is analytic in  $0 < |z| < 1$  and  $1 < |z| < \infty$ .

For  $0 < |z| < 1$ ,

$$\begin{aligned} f(z) &= \frac{1}{z(1+z^2)} = \frac{1}{z} \left( \frac{1}{1-(-z^2)} \right) \\ &= \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n z^{2n} = \sum_{n=0}^{\infty} (-1)^n z^{2n-1} \end{aligned}$$

and for  $1 < |z| < \infty$ ,

$$\begin{aligned} f(z) &= \frac{1}{z(1+z^2)} = \frac{1}{z^3} \left( \frac{1}{1-(-z^{-2})} \right) \\ &= \frac{1}{z^3} \sum_{n=0}^{\infty} (-1)^n z^{-2n} = \sum_{n=0}^{\infty} (-1)^n z^{-2n-3} \end{aligned}$$

■

6. Let

$$f(z) = \frac{z^2}{z^2 - 3z + 2}$$

Find the Laurent series of  $f(z)$  in each of the following domains:

(a)  $1 < |z| < 2$

(b)  $1 < |z - 3| < 2$

*Solution.* First, we write  $f(z)$  as a sum of partial fractions:

$$\frac{z^2}{z^2 - 3z + 2} = 1 + \frac{3z - 2}{(z - 2)(z - 1)} = 1 + \frac{4}{z - 2} - \frac{1}{z - 1}$$

In  $1 < |z| < 2$ ,

$$\begin{aligned} \frac{z^2}{z^2 - 3z + 2} &= 1 - \frac{2}{1 - z/2} - \frac{1}{z} \frac{1}{1 - 1/z} \\ &= 1 - 2 \sum_{n=0}^{\infty} 2^{-n} z^n - \frac{1}{z} \sum_{n=0}^{\infty} z^{-n} \\ &= -1 - \sum_{n=1}^{\infty} 2^{1-n} z^n - \sum_{n=1}^{\infty} z^{-n} \end{aligned}$$

In  $1 < |z - 3| < 2$ ,

$$\begin{aligned}
 \frac{z^2}{z^2 - 3z + 2} &= 1 + \frac{4}{(z - 3) + 1} - \frac{1}{2 + (z - 3)} \\
 &= 1 + \frac{4}{z - 3} \left( \frac{1}{1 + 1/(z - 3)} \right) - \frac{1}{2} \left( \frac{1}{1 + (z - 3)/2} \right) \\
 &= 1 + \frac{4}{z - 3} \sum_{n=0}^{\infty} (-1)^n (z - 3)^{-n} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n 2^{-n} (z - 3)^n \\
 &= \frac{1}{2} + 4 \sum_{n=1}^{\infty} (-1)^{n+1} (z - 3)^{-n} - \sum_{n=1}^{\infty} (-1)^n 2^{-n-1} (z - 3)^n
 \end{aligned}$$

■

7. Let

$$f(z) = \frac{z^2}{z^2 - z - 2}$$

Find the Laurent series of  $f(z)$  in each of the following domains:

(a)  $1 < |z| < 2$

(b)  $0 < |z - 2| < 1$

*Solution.* First, we write  $f(z)$  as a sum of partial fractions:

$$\frac{z^2}{z^2 - z - 2} = 1 + \frac{z + 2}{(z - 2)(z + 1)} = 1 + \frac{4}{3(z - 2)} - \frac{1}{3(z + 1)}$$

In  $1 < |z| < 2$ ,

$$\begin{aligned}
 \frac{z^2}{z^2 - z - 2} &= 1 - \frac{2}{3} \frac{1}{1 - z/2} - \frac{1}{3z} \frac{1}{1 + 1/z} \\
 &= 1 - \frac{2}{3} \sum_{n=0}^{\infty} 2^{-n} z^n - \frac{1}{3z} \sum_{n=0}^{\infty} (-1)^n z^{-n} \\
 &= \frac{1}{3} - \frac{1}{3} \sum_{n=1}^{\infty} 2^{1-n} z^n + \frac{1}{3} \sum_{n=1}^{\infty} (-1)^n z^{-n}
 \end{aligned}$$

In  $0 < |z - 2| < 1$ ,

$$\begin{aligned}
 \frac{z^2}{z^2 - z - 2} &= 1 + \frac{4}{3(z-2)} - \frac{1}{3} \frac{1}{3 + (z-2)} \\
 &= 1 + \frac{4}{3(z-2)} - \frac{1}{9} \frac{1}{1 + (z-2)/3} \\
 &= 1 + \frac{4}{3(z-2)} - \frac{1}{9} \sum_{n=0}^{\infty} (-1)^n 3^{-n} (z-2)^n \\
 &= \frac{8}{9} + \frac{4}{3(z-2)} + \sum_{n=1}^{\infty} (-1)^{n+1} 3^{-n-2} (z-2)^n
 \end{aligned}$$

■

8. Find the Laurent series of

$$\frac{1}{e^{z^2} - 1}$$

in  $z$  up to  $z^6$  and show the series converges in  $0 < |z| < \sqrt{2\pi}$ .

*Solution.* Let  $f(z) = 1/(e^z - 1)$ . Since

$$e^z - 1 = \sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!} = z \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

$e^z - 1$  has a zero at 0 of multiplicity one and hence  $f(z)$  has pole at 0 of order 1. So the Laurent series of  $f(z)$  is given by

$$f(z) = \sum_{n=-1}^{\infty} a_n z^n = \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \sum_{n \geq 4} a_n z^n$$

in  $0 < |z| < r$  for some  $r > 0$ .

Since  $(e^z - 1)f(z) = 1$ , we have

$$\begin{aligned}
 1 &= \left( a_{-1} + a_0 z + a_1 z^2 + a_2 z^3 + a_3 z^4 + \sum_{n \geq 5} a_{n-1} z^n \right) \\
 &\quad \left( 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + \sum_{n \geq 5} \frac{z^n}{(n+1)!} \right).
 \end{aligned}$$

Comparing the coefficients of  $1, z, z^2, z^3$  and  $z^4$  on both sides, we obtain

$$\begin{cases} a_{-1} = 1 \\ a_0 + \frac{a_{-1}}{2} = 0 \\ a_1 + \frac{a_0}{2} + \frac{a_{-1}}{6} = 0 \\ a_2 + \frac{a_1}{2} + \frac{a_0}{6} + \frac{a_{-1}}{24} = 0 \\ a_3 + \frac{a_2}{2} + \frac{a_1}{6} + \frac{a_0}{24} + \frac{a_{-1}}{120} = 0 \end{cases}$$

Solving it, we have  $a_{-1} = 1, a_0 = -1/2, a_1 = 1/12, a_2 = 0$  and  $a_3 = -1/720$ . Hence

$$f(z) = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} - \frac{z^3}{720} + \sum_{n \geq 4} a_n z^n$$

and

$$\frac{1}{e^{z^2} - 1} = f(z^2) = \frac{1}{z^2} - \frac{1}{2} + \frac{z^2}{12} - \frac{z^6}{720} + \sum_{n \geq 4} a_n z^{2n}.$$

Note that  $f(z)$  is analytic in  $\{z : e^z - 1 \neq 0\} = \{z \neq 2n\pi i\}$ . So it is analytic in  $0 < |z| < 2\pi$ . Therefore,  $f(z^2)$  is analytic in  $0 < |z^2| < 2\pi$ , i.e.,  $0 < |z| < \sqrt{2\pi}$ . So the series converges in  $0 < |z| < \sqrt{2\pi}$ . ■

## 4.2 Classification of singularities

1. For each of the following complex functions, do the following:

- find all its singularities in  $\mathbb{C}$ ;
- write the principal part of the function at each singularity;
- for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity;
- compute the residue of the function at each singularity.

(a)  $f(z) = \frac{1}{(\cos z)^2}$

*Solution.*  $f(z)$  is singular at  $\cos z = 0$ , i.e.,  $z = n\pi + \pi/2$ . Let  $w = z - n\pi - \pi/2$ . Then

$$\frac{1}{(\cos z)^2} = \frac{1}{(\cos(w + n\pi + \pi/2))^2} = \frac{1}{(\sin w)^2}.$$



Since  $\sin w$  has a zero of multiplicity one at  $w = 0$ ,  $f(z)$  has a pole of order 2 at  $z = n\pi + \pi/2$ . So

$$\frac{1}{(\sin w)^2} = \frac{a_{-2}}{w^2} + \frac{a_{-1}}{w} + \sum_{n \geq 0} a_n w^n.$$

Since

$$(\sin w)^2 = \left( \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!} \right)^2 = w^2 + \sum_{n=4}^{\infty} b_n w^n$$

we have

$$1 = \left( \frac{a_{-2}}{w^2} + \frac{a_{-1}}{w} + \sum_{n \geq 0} a_n w^n \right) \left( w^2 + \sum_{n=4}^{\infty} b_n w^n \right).$$

Comparing the coefficients of 1 and  $w$  on both sides, we obtain  $a_{-2} = 1$  and  $a_{-1} = 0$ . So the principal part of  $f(z)$  at  $z = n\pi + \pi/2$  is

$$\frac{1}{(z - n\pi - \pi/2)^2}$$

with residue 0. ■

(b)  $f(z) = (1 - z^3) \exp\left(\frac{1}{z}\right)$

*Solution.* Since  $e^z = \sum_{n=0}^{\infty} z^n / n!$ ,

$$\begin{aligned} (1 - z^3) \exp\left(\frac{1}{z}\right) &= (1 - z^3) \sum_{n=0}^{\infty} \frac{1}{n! z^n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n! z^n} - \sum_{n=0}^{\infty} \frac{1}{n! z^{n-3}} \\ &= \sum_{n=0}^{\infty} \frac{1}{n! z^n} - \sum_{n=4}^{\infty} \frac{1}{n! z^{n-3}} - \sum_{n=0}^3 \frac{z^{3-n}}{n!} \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n! z^n} - \sum_{n=1}^{\infty} \frac{1}{(n+3)! z^n} - \sum_{n=0}^3 \frac{z^{3-n}}{n!} \\ &= \sum_{n=1}^{\infty} \left( \frac{1}{n!} - \frac{1}{(n+3)!} \right) \frac{1}{z^n} + 1 - \sum_{n=0}^3 \frac{z^{3-n}}{n!} \end{aligned}$$

Therefore,  $f(z)$  has an essential singularity at  $z = 0$  with principal part

$$\sum_{n=1}^{\infty} \left( \frac{1}{n!} - \frac{1}{(n+3)!} \right) \frac{1}{z^n}$$

and residue

$$\operatorname{Res}_{z=0} f(z) = \frac{1}{1!} - \frac{1}{4!} = \frac{23}{24}.$$

(c)  $f(z) = \frac{\sin z}{z^{2010}}$

*Solution.* Since

$$\begin{aligned} \frac{\sin z}{z^{2010}} &= \frac{1}{z^{2010}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-2009}}{(2n+1)!} \\ &= \sum_{n=0}^{1004} \frac{(-1)^n z^{2n-2009}}{(2n+1)!} + \sum_{n=1005}^{\infty} \frac{(-1)^n z^{2n-2009}}{(2n+1)!} \end{aligned}$$

$f(z)$  has a pole of order 2009 at  $z = 0$  with principal part

$$\sum_{n=0}^{1004} \frac{(-1)^n z^{2n-2009}}{(2n+1)!}$$

and with residue

$$\operatorname{Res}_{z=0} \frac{\sin z}{z^{2010}} = \frac{(-1)^{1004}}{(2 \cdot 1004 + 1)!} = \frac{1}{2009!}.$$

(d)  $f(z) = \frac{e^z}{1 - z^2}$

*Solution.* Since  $1 - z^2 = (1 - z)(1 + z)$ ,  $f(z)$  has poles of order 1 at 1 and  $-1$ . Therefore,

$$\operatorname{Res}_{z=1} \frac{e^z}{1 - z^2} = \left. \frac{e^z}{(1 - z^2)'} \right|_{z=1} = -\frac{e}{2}$$

and

$$\operatorname{Res}_{z=-1} \frac{e^z}{1 - z^2} = \left. \frac{e^z}{(1 - z^2)'} \right|_{z=-1} = \frac{1}{2e}.$$

And the principal parts of  $f(z)$  at  $z = 1$  and  $z = -1$  are

$$-\frac{e}{2(z-1)} \text{ and } \frac{1}{2e(z+1)}$$

respectively.

(e)  $f(z) = (1 - z^2) \exp\left(\frac{1}{z}\right)$

*Solution.* The function has a singularity at 0 where

$$\begin{aligned}
 (1 - z^2) \exp\left(\frac{1}{z}\right) &= (1 - z^2) \sum_{n=0}^{\infty} \frac{1}{(n!)z^n} \\
 &= \sum_{n=0}^{\infty} \frac{1}{(n!)z^n} - \sum_{n=0}^{\infty} \frac{1}{(n!)z^{n-2}} \\
 &= 1 + \sum_{n=1}^{\infty} \frac{1}{(n!)z^n} - \sum_{n=3}^{\infty} \frac{1}{(n!)z^{n-2}} - (z^2 + z + \frac{1}{2}) \\
 &= -z^2 - z + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n!)z^n} - \sum_{n=1}^{\infty} \frac{1}{(n+2)!z^n} \\
 &= -z^2 - z + \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{1}{n!} - \frac{1}{(n+2)!} \right) z^{-n}
 \end{aligned}$$

So the principal part is

$$\sum_{n=1}^{\infty} \left( \frac{1}{n!} - \frac{1}{(n+2)!} \right) z^{-n}$$

the function has an essential singularity at 0 and

$$\operatorname{Res}_{z=0} f(z) = \frac{1}{1!} - \frac{1}{3!} = \frac{5}{6}$$

■

$$(f) \ f(z) = \frac{1}{(\sin z)^2}$$

*Solution.* The function has singularities at  $k\pi$  for  $k \in \mathbb{Z}$ . At  $z = k\pi$ , we let  $w = z - k\pi$  and then

$$\begin{aligned} \frac{1}{(\sin z)^2} &= \frac{1}{(\sin w)^2} = \left( \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!} \right)^{-1} \\ &= \frac{1}{w^2} \left( \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n+1)!} \right)^{-1} \\ &= \frac{1}{w^2} \left( 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} w^{2n}}{(2n+1)!} \right)^{-1} \\ &= \frac{1}{w^2} \sum_{m=0}^{\infty} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} w^{2n}}{(2n+1)!} \right)^m \\ &= \frac{1}{w^2} \left( 1 + \sum_{n=2}^{\infty} a_n w^n \right) \end{aligned}$$

So the principal part at  $k\pi$  is

$$\frac{1}{(z - k\pi)^2}$$

the function has a pole of order 2 at  $k\pi$  and

$$\operatorname{Res}_{z=k\pi} f(z) = 0$$

■

$$(g) \ f(z) = \frac{1 - \cos z}{z^2}$$

*Solution.* The function has a singularity at 0 where

$$\begin{aligned} \frac{1 - \cos z}{z^2} &= \frac{1}{z^2} \left( 1 - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \right) = \frac{1}{z^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^{2n}}{(2n)!} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^{2n-2}}{(2n)!} \end{aligned}$$

So the principal part is 0, the function has a removable singularity at 0 and

$$\operatorname{Res}_{z=0} f(z) = 0$$

■

$$(h) f(z) = \frac{e^z}{z(z-1)^2}$$

*Solution.* The function has two singularities at 0 and 1. At  $z = 0$ ,

$$\begin{aligned} \frac{e^z}{z(z-1)^2} &= \frac{1}{z} \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} (n+1)z^n \right) \\ &= \frac{1}{z} \left( 1 + \sum_{n=1}^{\infty} a_n z^n \right) \end{aligned}$$

So the principal part at 0 is  $1/z$ , the function has a pole of order 1 at 0 and

$$\operatorname{Res}_{z=0} f(z) = 1$$

At  $z = 1$ , we let  $w = z - 1$  and then

$$\begin{aligned} \frac{e^z}{z(z-1)^2} &= \frac{e^{1+w}}{(1+w)w^2} = \frac{e}{w^2} \left( \sum_{n=0}^{\infty} \frac{w^n}{n!} \right) \left( \sum_{n=0}^{\infty} (-1)^n w^n \right) \\ &= \frac{e}{w^2} \left( 1 + w + \sum_{n=2}^{\infty} \frac{w^n}{n!} \right) \left( 1 - w + \sum_{n=2}^{\infty} (-1)^n w^n \right) \\ &= \frac{e}{w^2} \left( 1 + \sum_{n=2}^{\infty} a_n w^n \right) \end{aligned}$$

So the principal part at 1 is

$$\frac{e}{(z-1)^2}$$

the function has a pole of order 2 at 1 and

$$\operatorname{Res}_{z=1} f(z) = 0$$

■

(i)  $f(z) = \tan z$

*Solution.* The function has singularities at  $\{\cos z = 0\} = \{z = k\pi + \pi/2 : k \in \mathbb{Z}\}$ . At  $z = k\pi + \pi/2$ , we let  $w = z - k\pi - \pi/2$  and then

$$\tan z = \tan \left( w + k\pi + \frac{\pi}{2} \right) = \tan \left( w + \frac{\pi}{2} \right) = -\frac{\cos w}{\sin w}$$

Since  $\sin w$  has a zero of multiplicity 1 at  $w = 0$ ,  $\tan z$  has a pole of order 1 at  $z = k\pi + \pi/2$ . Therefore

$$\operatorname{Res}_{z=k\pi+\pi/2} \tan z = \operatorname{Res}_{w=0} \left( -\frac{\cos w}{\sin w} \right) = - \left. \frac{\cos w}{(\sin w)'} \right|_{w=0} = -1$$

and the principal part of  $\tan z$  at  $z = k\pi + \pi/2$  is

$$-\frac{1}{w} = -\frac{1}{z - k\pi - \pi/2}.$$

■

(j)  $f(z) = (1 - z^2) \sin \left( \frac{1}{z} \right)$

*Solution.* The function has a singularity at 0 where

$$\begin{aligned} (1 - z^2) \sin \left( \frac{1}{z} \right) &= (1 - z^2) \sum_{n=0}^{\infty} \frac{(-1)^n}{((2n+1)!) z^{2n+1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{((2n+1)!) z^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{((2n+1)!) z^{2n-1}} \\ &= -z + \sum_{n=0}^{\infty} \frac{(-1)^n}{((2n+1)!) z^{2n+1}} - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)! z^{2n-1}} \\ &= -z + \sum_{n=0}^{\infty} \frac{(-1)^n}{((2n+1)!) z^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{((2n+3)!) z^{2n+1}} \\ &= -z + \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{(2n+1)!} + \frac{1}{(2n+3)!} \right) z^{-2n-1} \end{aligned}$$

So the principal part is

$$\sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{(2n+1)!} + \frac{1}{(2n+3)!} \right) z^{-2n-1}.$$

Therefore, the function has an essential singularity at 0 and

$$\operatorname{Res}_{z=0} f(z) = \frac{1}{1!} + \frac{1}{3!} = \frac{7}{6}.$$

■

$$(k) f(z) = \frac{e^z}{z^{2011}}$$

*Solution.* The function has a singularity at 0 where

$$\begin{aligned} \frac{e^z}{z^{2011}} &= \frac{1}{z^{2011}} \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{z^{n-2011}}{n!} = \sum_{n=0}^{2010} \frac{z^{n-2011}}{n!} + \sum_{n=2011}^{\infty} \frac{z^{n-2011}}{n!} \end{aligned}$$

Therefore, the principal part of  $f(z)$  at  $z = 0$  is

$$\sum_{n=0}^{2010} \frac{z^{n-2011}}{n!}$$

and  $f(z)$  has a pole of order 2011 and residue

$$\operatorname{Res}_{z=0} f(z) = \frac{1}{2010!}$$

at  $z = 0$ . ■

$$(l) f(z) = \frac{\cos z}{z^2 - z^3}$$

*Solution.* The function has singularities at  $\{z^2 - z^3 = 0\} = \{z = 0, 1\}$ . At  $z = 0$ ,  $z^2 - z^3$  has a zero of multiplicity 2 and hence  $f(z)$  has a pole of order 2. Suppose that the Laurent series of  $f(z)$  at  $z = 0$  is given by

$$\frac{\cos z}{z^2 - z^3} = \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \sum_{n \geq 0} a_n z^n.$$

Hence

$$(z^2 - z^3) \left( \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \sum_{n \geq 0} a_n z^n \right) = \cos z = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}.$$

Comparing the coefficients of 1 and  $z$  on both sides, we obtain that  $a_{-2} = 1$  and  $a_{-1} - a_{-2} = 0$  and hence  $a_{-1} = a_{-2} = 1$ . So the principal part of  $f(z)$  at  $z = 0$  is

$$\frac{1}{z^2} + \frac{1}{z}$$

with residue

$$\operatorname{Res}_{z=0} f(z) = 1.$$

At  $z = 1$ ,  $z^2 - z^3$  has a zero of multiplicity 1 and hence  $f(z)$  has a pole of order 1. Hence

$$\operatorname{Res}_{z=1} \frac{\cos z}{z^2 - z^3} = \frac{\cos z}{(z^2 - z^3)'} \Big|_{z=1} = -\cos(1)$$

and the principal part of  $f(z)$  at  $z = 1$  is

$$-\frac{\cos(1)}{z - 1}.$$

■

### 4.3 Applications of residues

1. Compute the integral

$$\int_0^\pi \frac{d\theta}{2 - \cos \theta}.$$

*Solution.* Since  $1/(2 - \cos \theta)$  is even,

$$\int_0^\pi \frac{d\theta}{2 - \cos \theta} = \frac{1}{2} \int_{-\pi}^\pi \frac{d\theta}{2 - \cos \theta}.$$

Let  $z = e^{i\theta}$ . Then  $\cos \theta = (z + z^{-1})/2$  and  $d\theta = -iz^{-1}dz$ . Hence

$$\begin{aligned} \int_0^\pi \frac{d\theta}{2 - \cos \theta} &= \frac{1}{2} \int_{-\pi}^\pi \frac{d\theta}{2 - \cos \theta} \\ &= \int_{|z|=1} \frac{-idz}{2z(2 - (z + z^{-1})/2)} \\ &= i \int_{|z|=1} \frac{dz}{z^2 - 4z + 1}. \end{aligned}$$

The function

$$\frac{1}{z^2 - 4z + 1} = \frac{1}{(z - 2 - \sqrt{3})(z - 2 + \sqrt{3})}$$

has a singularity in  $|z| < 1$  at  $z = 2 - \sqrt{3}$ . Therefore,

$$\begin{aligned} \int_{|z|=1} \frac{dz}{z^2 - 4z + 1} &= 2\pi i \operatorname{Res}_{z=2-\sqrt{3}} \frac{1}{z^2 - 4z + 1} \\ &= 2\pi i \frac{1}{(z^2 - 4z + 1)'} \Big|_{z=2-\sqrt{3}} = -\frac{\pi i}{\sqrt{3}}. \end{aligned}$$



Therefore,

$$\int_0^\pi \frac{d\theta}{2 - \cos \theta} = \frac{\pi}{\sqrt{3}}$$

■

2. Calculate the integral

$$\int_0^\pi \frac{d\theta}{a + b \cos \theta} \quad a^2 > b^2.$$

Answer:  $\frac{\pi}{\sqrt{a^2 - b^2}}$

3. Evaluate the contour integral of the following functions around the circle  $|z| = 2011$  oriented counterclockwise:

(a)  $\frac{1}{\sin z}$ ;

(b)  $\frac{1}{e^{2z} - e^z}$ .

*Solution.* (a)  $f(z) = 1/\sin z$  is analytic in  $\{z \neq n\pi : n \in \mathbb{Z}\}$ . It has a pole of order one at  $n\pi$  since  $(\sin z)'|_{z=n\pi} = \cos(n\pi) = (-1)^n \neq 0$ . So

$$\operatorname{Res}_{z=n\pi} \frac{1}{\sin z} = \frac{1}{\cos(n\pi)} = (-1)^n.$$

Therefore,

$$\begin{aligned} \int_{|z|=2011} \frac{dz}{\sin z} &= 2\pi i \sum_{|n\pi| < 2011} \operatorname{Res}_{z=n\pi} \frac{1}{\sin z} \\ &= 2\pi i \sum_{|n| \leq 640} (-1)^n = 2\pi i. \end{aligned}$$

(b)  $f(z) = 1/(e^{2z} - e^z)$  is analytic in

$$\{e^{2z} - e^z \neq 0\} = \{e^z \neq 1\} = \{z \neq 2n\pi i : n \in \mathbb{Z}\}.$$

Since  $(e^{2z} - e^z)'|_{z=2n\pi i} = 1 \neq 0$ ,  $f(z)$  has a pole of order one at  $2n\pi i$ . So

$$\operatorname{Res}_{z=2n\pi i} \frac{1}{e^{2z} - e^z} = \frac{1}{2e^{2z} - e^z} \Big|_{z=2n\pi i} = 1.$$

Therefore,

$$\begin{aligned} \int_{|z|=2011} \frac{dz}{e^{2z} - e^z} &= 2\pi i \sum_{|2n\pi i| < 2011} \operatorname{Res}_{z=2n\pi i} \frac{1}{e^{2z} - e^z} \\ &= 2\pi i \sum_{|2n\pi i| < 2011} 1 = 2\pi i \sum_{|n| \leq 320} 1 = 1282\pi i. \end{aligned}$$

■

4. Let

$$f(z) = (z - a_1)(z - a_2) \dots (z - a_n)$$

be a complex polynomial with  $n \geq 2$  distinct roots  $a_1, a_2, \dots, a_n$ .

(a) Prove that

$$\int_{|z|=R} \frac{dz}{f(z)} = 2\pi i \sum_{k=1}^n \frac{1}{\prod_{j \neq k} (a_k - a_j)}$$

for  $R > |a_k|$  ( $k = 1, 2, \dots, n$ ).

(b) Use (a) and Cauchy Integral Theorem to prove that

$$\sum_{k=1}^n \frac{1}{\prod_{j \neq k} (a_k - a_j)} = 0$$

for all distinct complex numbers  $a_1, a_2, \dots, a_n$ .

*Proof.* By Residue theorem,

$$\int_{|z|=R} \frac{dz}{f(z)} = 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=a_k} \frac{1}{f(z)}.$$

At each  $a_k$ ,  $1/f(z)$  has a pole of order one and

$$\operatorname{Res}_{z=a_k} \frac{1}{f(z)} = \operatorname{Res}_{z=a_k} \frac{\left( \prod_{j \neq k} (z - a_j) \right)^{-1}}{z - a_k} = \frac{1}{\prod_{j \neq k} (a_k - a_j)}.$$

Therefore,

$$\int_{|z|=R} \frac{dz}{f(z)} = 2\pi i \sum_{k=1}^n \frac{1}{\prod_{j \neq k} (a_k - a_j)}.$$

Since  $\deg(f(z)) = n \geq 2$ ,

$$\int_{|z|=R} \frac{dz}{f(z)} = 0$$

by Problem 18 in Section 3.2. Therefore,

$$\sum_{k=1}^n \frac{1}{\prod_{j \neq k} (a_k - a_j)} = 0.$$

■

5. Use Cauchy Integral Theorem or Residue Theorem to show that

$$\frac{1}{2\pi i} \int_{C_N} \frac{dz}{z^2 \sin z} = \frac{1}{6} + 2 \sum_{n=1}^N \frac{(-1)^n}{n^2 \pi^2}$$

and conclude that

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

*Solution.* The function  $f(z) = 1/(z^2 \sin z)$  has singularities at  $z = n\pi$  for  $n \in \mathbb{Z}$ . So

$$\frac{1}{2\pi i} \int_{C_N} \frac{dz}{z^2 \sin z} = \sum_{n=-N}^n \operatorname{Res}_{z=n\pi} \frac{1}{z^2 \sin z}$$

by Residue Theorem.

At  $z = n\pi$  for a nonzero integer  $n$ ,

$$z^2 \Big|_{n\pi} \neq 0 \quad \text{and} \quad (\sin z)' \Big|_{n\pi} \neq 0.$$

Therefore,  $1/(z^2 \sin z)$  has a pole of order 1 at  $n\pi$  for  $n \neq 0$ . It follows that

$$\operatorname{Res}_{z=n\pi} \frac{1}{z^2 \sin z} = \frac{1}{z^2 (\sin z)'} \Big|_{z=n\pi} = \frac{1}{n^2 \pi^2 \cos(n\pi)} = \frac{(-1)^n}{n^2 \pi^2}$$

for  $n \neq 0$ .

At  $z = 0$ ,  $z^2 \sin z$  has a zero multiplicity 3 and hence  $1/(z^2 \sin z)$  has a pole of order 3. Suppose that the Laurent series of  $f(z)$  at  $z = 0$  is given by

$$\frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \sum_{n \geq 0} a_n z^n.$$

Then

$$\begin{aligned} & z^2 \sin z \left( \frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + \sum_{n \geq 0} a_n z^n \right) \\ &= \left( 1 - \frac{z^2}{3!} + \sum_{n \geq 3} b_n z^n \right) \left( a_{-3} + a_{-2}z + a_{-1}z^2 + \sum_{n \geq 3} a_n z^n \right) = 1. \end{aligned}$$

Comparing the coefficients of 1,  $z$  and  $z^2$  on both sides, we have

$$\begin{cases} a_{-3} = 1 \\ a_{-2} = 0 \\ a_{-1} - \frac{a_{-3}}{6} = 0 \end{cases}$$

Solving the equation, we obtain  $a_{-1} = 1/6$ ,  $a_{-2} = 0$  and  $a_{-3} = 1$ . So

$$\operatorname{Res}_{z=0} \frac{1}{z^2 \sin z} = \frac{1}{6}.$$

Therefore,

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_N} \frac{dz}{z^2 \sin z} &= \sum_{n=-N}^n \operatorname{Res}_{z=n\pi} \frac{1}{z^2 \sin z} \\ &= \frac{1}{6} + \sum_{n=-N}^{n=-1} \frac{(-1)^n}{n^2 \pi^2} + \sum_{n=1}^{n=N} \frac{(-1)^n}{n^2 \pi^2}. \end{aligned}$$

We observe that

$$\frac{(-1)^n}{n^2 \pi^2} = \frac{(-1)^{-n}}{(-n)^2 \pi^2}$$

and hence we obtain

$$\frac{1}{2\pi i} \int_{C_N} \frac{dz}{z^2 \sin z} = \frac{1}{6} + 2 \sum_{n=1}^N \frac{(-1)^n}{n^2 \pi^2}.$$

By Problem 11 in section 3.2, we have

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi i} \int_{C_N} \frac{dz}{z^2 \sin z} = 0$$

Consequently,

$$\frac{1}{6} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 \pi^2} = 0$$

That is,

$$\frac{\pi^2}{12} = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$



### 4.3.1 Improper integrals

1. Compute the integral  $\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1} dx$

*Solution.* since  $e^{ix} = \cos x + i \sin x$ ,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1} dx = \operatorname{Re} \left( \int_{-\infty}^{\infty} \frac{e^{ix}}{x^4 + x^2 + 1} dx \right)$$

Actually, we have

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1} dx = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^4 + x^2 + 1} dx$$

in this case since  $\sin x / (x^4 + x^2 + 1)$  is odd.

Consider the contour integral of  $e^{iz} / (z^4 + z^2 + 1)$  along the path  $L_R = [-R, R]$  and  $C_R = \{|z| = R, \operatorname{Im}(z) \geq 0\}$ , oriented counterclockwise. By CIT or residue theorem, we have

$$\begin{aligned} & \int_{L_R} \frac{e^{iz}}{z^4 + z^2 + 1} dz + \int_{C_R} \frac{e^{iz}}{z^4 + z^2 + 1} dz \\ &= 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} \frac{e^{iz}}{z^4 + z^2 + 1} \end{aligned}$$

where  $z_1, z_2, \dots, z_n$  are the singularities of  $e^{iz} / (z^4 + z^2 + 1)$  inside the region  $\{|z| < R, \operatorname{Im}(z) > 0\}$ .

We find the singularities of  $e^{iz} / (z^4 + z^2 + 1)$  by solving  $z^4 + z^2 + 1 = 0$ : we observe that  $(z^2 - 1)(z^4 + z^2 + 1) = z^6 - 1$ . So the function has four singularities  $\pm e^{\pi i/3}$  and  $\pm e^{2\pi i/3}$ . Two of them  $e^{\pi i/3}$  and  $e^{2\pi i/3}$  lie above the real axis. Therefore,

$$\begin{aligned} & \int_{L_R} \frac{e^{iz}}{z^4 + z^2 + 1} dz + \int_{C_R} \frac{e^{iz}}{z^4 + z^2 + 1} dz \\ &= 2\pi i \left( \operatorname{Res}_{z=e^{\pi i/3}} \frac{e^{iz}}{z^4 + z^2 + 1} + \operatorname{Res}_{z=e^{2\pi i/3}} \frac{e^{iz}}{z^4 + z^2 + 1} \right). \end{aligned}$$

Since all zeros of  $z^4 + z^2 + 1$  have multiplicity one, all poles of

$$e^{iz} / (z^4 + z^2 + 1)$$

have order one. Therefore,

$$\operatorname{Res}_{z=e^{\pi i/3}} \frac{e^{iz}}{z^4 + z^2 + 1} = \frac{e^{iz}}{(z^4 + z^2 + 1)'} \Big|_{z=e^{\pi i/3}} = \frac{\exp((- \sqrt{3} + i)/2)}{\sqrt{3}i - 3}$$

and

$$\operatorname{Res}_{z=e^{2\pi i/3}} \frac{e^{iz}}{z^4 + z^2 + 1} = \frac{e^{iz}}{(z^4 + z^2 + 1)'} \Big|_{z=e^{2\pi i/3}} = \frac{\exp((- \sqrt{3} - i)/2)}{\sqrt{3}i + 3}.$$

Hence

$$\int_{L_R} \frac{e^{iz}}{z^4 + z^2 + 1} dz + \int_{C_R} \frac{e^{iz}}{z^4 + z^2 + 1} dz = \frac{\pi}{3} \left( \sqrt{3} \cos \left( \frac{1}{2} \right) + 3 \sin \left( \frac{1}{2} \right) \right).$$

For  $z$  lying on  $C_R$ ,  $y = \operatorname{Im}(z) \geq 0$  and hence  $|e^{iz}| = e^{-y} \leq 1$ . Hence

$$\left| \frac{e^{iz}}{z^4 + z^2 + 1} \right| \leq \frac{1}{R^4 - R^2 - 1}$$

and it follows that

$$\left| \int_{C_R} \frac{e^{iz}}{z^4 + z^2 + 1} dz \right| \leq \frac{\pi R}{R^4 - R^2 - 1}$$

Since

$$\lim_{R \rightarrow \infty} \frac{\pi R}{R^4 - R^2 - 1} = 0,$$

we conclude that

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{iz}}{z^4 + z^2 + 1} dz = 0.$$

Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1} dx &= \int_{-\infty}^{\infty} \frac{e^{ix}}{x^4 + x^2 + 1} dx \\ &= \lim_{R \rightarrow \infty} \int_{L_R} \frac{e^{iz}}{z^4 + z^2 + 1} dz \\ &= \frac{\pi}{3} \left( \sqrt{3} \cos \left( \frac{1}{2} \right) + 3 \sin \left( \frac{1}{2} \right) \right). \end{aligned}$$

■

2. Compute the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x + 2} dx.$$

*Solution.* since  $e^{ix} = \cos x + i \sin x$ ,

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x^2 + 2} dx = \operatorname{Im} \left( \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 2x^2 + 2} dx \right)$$

Consider the contour integral of  $e^{iz}/(z^2 + 2z^2 + 2)$  along the path  $L_R = [-R, R]$  and  $C_R = \{|z| = R, \operatorname{Im}(z) \geq 0\}$ , oriented counterclockwise.

Since  $e^{iz}/(z^2 + 2z + 2)$  has two isolated singularities at  $-1 \pm i$  with  $-1 + i$  lying inside the curve  $L_R \cup C_R$ , we have

$$\begin{aligned} & \int_{L_R} \frac{e^{iz}}{z^2 + 2z^2 + 2} dz + \int_{C_R} \frac{e^{iz}}{z^2 + 2z^2 + 2} dz \\ &= 2\pi i \operatorname{Res}_{z=-1+i} \frac{e^{iz}}{z^2 + 2z + 2} \\ &= 2\pi i \left. \frac{e^{iz}}{(z^2 + 2z + 2)'} \right|_{z=-1+i} \\ &= \frac{2\pi i \exp(-i - 1)}{2i} = \frac{\pi}{e} (\cos(1) - i \sin(1)) \end{aligned}$$

by Cauchy Integral Theorem or residue theorem.

For  $z$  lying on  $C_R$ ,  $y = \operatorname{Im}(z) \geq 0$  and hence  $|e^{iz}| = e^{-y} \leq 1$ . Hence

$$\left| \frac{e^{iz}}{z^2 + 2z^2 + 2} \right| \leq \frac{1}{R^2 - 2R^2 - 2}$$

and it follows that

$$\left| \int_{C_R} \frac{e^{iz}}{z^2 + 2z^2 + 2} dz \right| \leq \frac{\pi R}{R^2 - 2R^2 - 2}$$

Since

$$\lim_{R \rightarrow \infty} \frac{\pi R}{R^2 - 2R^2 - 2} = 0,$$

we conclude that

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{iz}}{z^2 + 2z^2 + 2} dz = 0.$$

Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 2x^2 + 2} dx &= \operatorname{Im} \left( \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 2x^2 + 2} dx \right) \\ &= \operatorname{Im} \left( \lim_{R \rightarrow \infty} \int_{L_R} \frac{e^{iz}}{z^2 + 2z^2 + 2} dz \right) \\ &= -\frac{\pi}{e} \sin(1). \end{aligned}$$

■

# Bibliography

- [1] [Brown, J. W. & Churchill, R. V. \(2009\). \*Complex Variables and Applications\*. 8th Edition. New York: McGraw-Hill Higher Education.](#)
- [2] [Needham, T. \(1997\). \*Visual Complex Analysis\*. New York: The Clarendon Press, Oxford University Press.](#)
- [3] [Spiegel, M. R. \(1981\). \*Theory and problems of Complex variables\*. Singapore: McGraw-Hill.](#)